Topp-Leone Alpha Power Family of Distributions

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Abstract

In this paper, Topp-Leone alpha power family of distributions is presented based on the method called alpha power transformation which is suggested by Mahdavi and Kundu (2016). The Topp-Leone alpha power family of distributions is rich and flexible for modeling data. Some statistical properties of the proposed family are studied including quantile function, moments and moment generating function, mean residual life, and order statistics. Also, the results of the paper are specialized to Topp-Leone alpha power Weibull distribution. Some of its statistical properties are obtained. Moreover, the maximum likelihood estimators of the unknown parameters of the Topp-Leone alpha power Weibull distribution are obtained. A simulation study is conducted to evaluate the performance of the maximum likelihood estimators of the parameters of the Topp-Leone alpha power Weibull distribution. Finally, a real data set is applied, and the results show that the Topp-Leone alpha power Weibull distribution is more appropriate as compared to the Topp-Leone exponential, alpha power Weibull, and Weibull distributions.

Keywords: *Topp-Leone family, Alpha power transformation, Weibull distribution, Maximum likelihood estimation.*

1. Introduction

Several researchers are interested in establishing new generalized classes of univariate continuous distributions by modifying a baseline distribution with one or more parameters to generate new distributions. These extended distributions provide greater flexibility in certain applications to real data. Various methods of generating new statistical distributions were presented in the literature. [For more details see Marshall and Olkin (1997), Eugene *et al.* (2002), Cordeiro and Castro (2011), Alzaatreh *et al.* (2013), Lee *et al.* (2013), and Jones (2015)].

Sangsanit and Bodhisuwan (2016) introduced the Topp-Leone family of distributions. The *cumulative distribution function* (cdf) and *probability density function* (pdf) of the Topp-Leone family are given, respectively, by

$$G_{TL}(x;\theta) = \left[1-\left(1-G(x)\right)^2\right]^{\theta},\ x \in \mathbb{R};\ \theta > 0, \tag{1}$$

and

$$g_{TL}(x;\theta) = 2\theta g(x)[1-G(x)]\left[1-\left(1-G(x)\right)^2\right]^{\theta-1}. \tag{2}$$

Several Topp-Leone families of distributions can be generated by specifying the cdf, $G(x)$. Many authors studied the Topp-Leone family, for example the Topp-Leone Burr-XII distribution by Reyad and Othman (2017) and Topp-Leone exponentiated power Lindley by Aryuyuen (2018). Also, Reyad *et al*. (2019) presented the Topp-Leone generalized inverted Kumaraswamy distribution.

Mahdavi and Kundu (2016) proposed a method for introducing new lifetime distributions named as *alpha power transformation* (APT). If $F(x)$ and $f(x)$ are the cdf and pdf of a random variable *X*, then the cdf of APT is given by

$$G_{APT}(x) = \frac{\alpha^{F(x)}-1}{\alpha-1}, \qquad \alpha > 0;\ \alpha \neq 1, \tag{3}$$

and the corresponding pdf as

$$g_{APT}(x) = \frac{\ln\alpha}{\alpha-1} f(x)\alpha^{F(x)},\ x \in \mathbb{R}, \alpha \neq 1, \tag{4}$$

where α is the shape parameter.

The rest of the paper is organized as follows: In Section 2, *Topp-Leone Alpha power* (TLAP) family is presented. Useful expansions of TLAP family are derived in Section 3. In Section 4, general expressions for some statistical properties of the general family are obtained. The *Topp-Leone alpha power Weibull* (TLAW) distribution is introduced in Section 5. In Section 6, a simulation study is conducted to evaluate the performance of the estimators. Also, a real data set is analyzed in Section 7. Finally, the conclusion is given in Section 8.

2. The Topp-Leone Alpha Power Family

Substituting the cdf of the APT family defined in (3) as the baseline cdf in (1). Then, the cdf of the TLAP family is

$$G_{TLAP}(x;\Theta)=\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^\theta, \ \alpha,\theta>0; \ \ \alpha\neq 1, x\in\mathbb{R}, \quad (5)$$

where $\Theta=(\theta,\alpha)$.

The pdf can be written as

$$g_{TLAP}(x;\Theta)=\frac{2\theta\ln\alpha}{\alpha-1}f(x)\alpha^{F(x)}\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^{\theta-1}. \quad (6)$$

The *survival function* (sf), $S_{TLAP}(x;\Theta)$, *hazard rate function* (hrf), $h_{TLAP}(x;\Theta)$, and *reversed hazard rate function* (rhrf), $rh_{TLAP}(x;\Theta)$ are, respectively, given by

$$S_{TLAP}(x;\Theta)=1-\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^\theta, \ \ \alpha\neq 1, \quad (7)$$

$$h_{TLAP}(x;\Theta)=\frac{\frac{2\theta\ln\alpha}{\alpha-1}f(x)\alpha^{F(x)}\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^{\theta-1}}{1-\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^\theta}, \ \alpha\neq 1, \quad (8)$$

and

$$rh_{TLAP}(x;\Theta)=\frac{\frac{2\theta\ln\alpha}{\alpha-1}f(x)\alpha^{F(x)}\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^{\theta-1}}{\left[1-\left(1-\frac{\alpha^{F(x)}-1}{\alpha-1}\right)^2\right]^{\theta}},\ \alpha\neq 1. \quad (9)$$

3. Expansions of the Topp-Leone Alpha Power Family

A useful linear representation for the pdf and cdf of the TLAP family are obtained using the binomial expansion and power series representation which are given below

$$(1-z)^n=\sum_{j=0}^{n}\binom{n}{j}(-1)^j z^j\ ,\ \alpha^{v}=\sum_{k=0}^{\infty}\frac{(v\ln\alpha)^k}{k!}\ . \quad (10)$$

Thus, the cdf in (5) can be written as

$$G_{TLAP}(x;\Theta)=\sum A_{j_1,j_2,j_3,j_4}\left(F(x)\right)^{j_4}, \quad (11)$$

where

$$\sum\ =\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{2j_1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{\infty}$$

and

$$A_{j_1,j_2,j_3,j_4}=\binom{\theta}{j_1}\binom{2j_1}{j_2}\binom{j_2}{j_3}\frac{(-1)^{j_1+2j_2+j_3}(j_3\ln\alpha)^{j_4}}{(\alpha-1)^{j_2}j_4!}.$$

Hence, the pdf of the TLAP family can be written as follows:

$$g_{TLAP}(x;\Theta)=\sum\nolimits_{*}D_{k_1,k_2,k_3,k_4}f(x)\left(F(x)\right)^{k_4}, \quad (12)$$

where

$$\sum\nolimits_{*}\ =\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{2k_1+1}\sum_{k_3=0}^{k_2}\sum_{k_4=0}^{\infty}$$

and

$$D_{k_1,k_2,k_3,k_4}=\binom{\theta-1}{k_1}\binom{2k_1+1}{k_2}\binom{k_2}{k_3}\frac{2\theta(-1)^{k_1+2k_2+k_3}(k_3+1)^{k_4}(\ln\alpha)^{k_4+1}}{(\alpha-1)^{k_2+1}k_4!}.$$

4. Some Statistical Properties

In this section, some statistical properties of the TLAP family including quantile function, moments and moment generating function, mean residual life, and order statistics are presented.

4.1 The quantile function

The quantile function of TLAP random variable X is given by

$$\ln\left\{(\alpha-1)\left[1-\left(1-U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]\right\}-F(x)\ln(\alpha)=0. \qquad (13)$$

4.2 Moments and moment generating function

Let $X \sim TLAP(x;\Theta)$, then using (12), the r^{th} moment of X is obtained as follows:

$$\mu'_{r(TLAP)}=\int_{-\infty}^{\infty}\sum\nolimits_{*}D_{k_1,k_2,k_3,k_4}x^r f(x)\big(F(x)\big)^{k_4}\,dx$$

$$=\sum\nolimits_{*}D_{k_1,k_2,k_3,k_4}Q_{r,k_4}(x), \qquad (14)$$

where

$$Q_{r,k_4}(x)=\int_{-\infty}^{\infty}x^r f(x)\big(F(x)\big)^{k_4}dx.$$

The moment generating function is

$$M_{x(TLAP)}(t)=\int_{-\infty}^{\infty}e^{tx}g(x;\Theta)dx\ ,$$

where

$$e^{tx}=\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}\ .$$

Then,

$$M_{x(TLAP)}(t)=\int_{-\infty}^{\infty}\sum\nolimits_{r=0}^{\infty}\frac{t^r}{r!}x^r g(x;\Theta)dx$$

$$=\sum\nolimits_{r=0}^{\infty}\sum\nolimits_{*}\frac{t^r}{r!}D_{k_1,k_2,k_3,k_4}Q_{r,k_4}(x). \qquad (15)$$

4.3 Residual life and reversed residual life

The residual lifetime function of TLAP random variable X is given by

$$R_{TLAP}(t) = \frac{S(x+t)}{S(t)}$$

$$= \frac{1-\left[1-\left(1-\frac{\alpha^{F(x+t)}-1}{\alpha-1}\right)^2\right]^\theta}{1-\left[1-\left(1-\frac{\alpha^{F(t)}-1}{\alpha-1}\right)^2\right]^\theta}. \quad (16)$$

The reversed residual lifetime function of TLAP random variable X is

$$\bar{R}_{TLAP}(t) = \frac{S(x-t)}{S(t)}$$

$$= \frac{1-\left[1-\left(1-\frac{\alpha^{F(x-t)}-1}{\alpha-1}\right)^2\right]^\theta}{1-\left[1-\left(1-\frac{\alpha^{F(t)}-1}{\alpha-1}\right)^2\right]^\theta}. \quad (17)$$

4.4 Order statistics

Considering that $X_1, X_2, \dots, X_n$ is a random sample of size n from TLAP family and $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ is the corresponding order statistics. Then, the pdf of the r^{th} order statistic is given by

$$g_{TLAP}(x_{(r)}) = \frac{g_{TLAP}(x_{(r)};\Theta)}{\boldsymbol{B}(r,n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[G_{TLAP}(x_{(r)};\Theta)\right]^{i+r-1}, x_{(r)} > 0, \quad (18)$$

where $\boldsymbol{B}(.,.)$denotes the beta function.

The pdf of the smallest order statistics $x_{(1)}$ and the largest order statistics $x_{(n)}$ are given below

$$g_{TLAP}(x_{(1)}) = n g_{TLAP}(x_{(1)};\Theta) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[G_{TLAP}(x_{(1)};\Theta)\right]^i, \ x_{(1)} > 0, \quad (19)$$

and

$$g_{TLAP}(x_{(n)}) = n g_{TLAP}(x_{(n)};\Theta) \left[G_{TLAP}(x_{(n)};\Theta)\right]^{n-1}, \ x_{(n)} > 0. \quad (20)$$

5. The Topp-Leone Alpha Power Weibull Distribution

In this section, the TLAP family is specialized to the Weibull distribution. The cdf and pdf of the Weibull distribution are given respectively by

$$F_W(x) = 1 - e^{-(x/\beta)^\lambda}, \tag{21}$$

and

$$f_W(x) = \frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1} e^{-(x/\beta)^\lambda},\ x > 0;\ \lambda, \beta > 0\,. \tag{22}$$

By substituting the cdf and pdf of the Weibull distribution, respectively, in (5) and (6). Hence, one can obtain the cdf and the pdf of the TLAW distribution as follows:

$$G_{TLAW}(x;\Omega) = \left[1 - \left(1 - \frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^\lambda}} - 1}{\alpha - 1}\right)^2\right]^\theta, \tag{23}$$

and

$$g_{TLAW}(x;\Omega) = \frac{2\theta\lambda\ln\alpha}{\beta(\alpha-1)}\left(\frac{x}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x}{\beta}\right)^\lambda} \alpha^{1-e^{-\left(\frac{x}{\beta}\right)^\lambda}} \left(1 - \frac{\alpha^{1-e^{-(x/\beta)^\lambda}} - 1}{\alpha - 1}\right)$$

$$\times \left[1 - \left(1 - \frac{\alpha^{1-e^{-(x/\beta)^\lambda}} - 1}{\alpha - 1}\right)^2\right]^{\theta-1}, x > 0;\ \theta, \alpha, \lambda, \beta > 0, \alpha \neq 1, \tag{24}$$

where $\Omega = (\theta, \alpha, \lambda, \beta)$.

The sf and hrf are, respectively, given by

$$S_{TLAW}(x;\Omega) = 1 - \left[1 - \left(1 - \frac{\alpha^{1-e^{-(x/\beta)^\lambda}} - 1}{\alpha - 1}\right)^2\right]^\theta,\ \ \alpha \neq 1, \tag{25}$$

and

$$h_{TLAW}(x;\Omega)=\frac{\frac{2\theta\lambda\ln\alpha}{\beta(\alpha-1)}\left(\frac{x}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta-1}}{1-\left[1-\left(1-\frac{\alpha^{1-e^{-(x/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}}. \tag{26}$$

Some sub-models of TLAW distribution (24) are given in Table 1 and the plots for the pdf and hrf of the TLAW distribution are displayed in Figure 1.

Table 1: Sub-models of TLAW distribution

θ	α	λ	β	cdf	Sub-model
—	—	—	1	$\left(1-\left(1-\frac{\alpha^{1-e^{-x^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}$	Topp-Leone alpha power one parameter Weibull distribution
—	—	1	—	$\left(1-\left(1-\frac{\alpha^{1-e^{-\frac{x}{\beta}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}$	Topp-Leone alpha power exponential distribution [See Rady *et al.* (2021)]
—	1	—	—	$\left(1-\left(e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right)^{2}\right)^{\theta}$	Topp-Leone Weibull distribution [See Tuoyo *et al*. (2021)]
—	—	2	—	$\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{2}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}$	Topp-Leone alpha power Rayleigh distribution
—	1	2	—	$\left(1-\left(e^{-\left(\frac{x}{\beta}\right)^{2}}\right)^{2}\right)^{\theta}$	Topp-Leone Rayleigh distribution [See Olayode (2019)]
—	1	1	—	$\left(1-\left(e^{-\frac{x}{\beta}}\right)^{2}\right)^{\theta}$	Topp-Leone exponential distribution [See Al-Shomrani *et al*. (2016)]

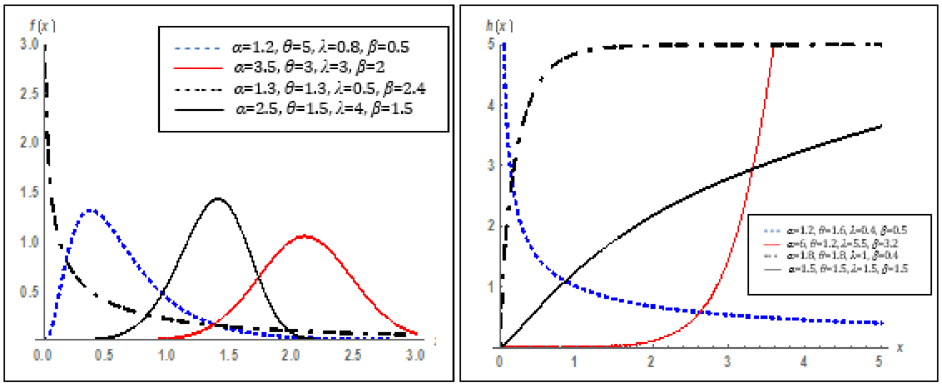

Figure 1: Different plots of pdf and hrf of the TLAW distribution

The plots of Figure 1 indicate that the TLAW density can be decreasing, unimodal, right skewed and left skewed. Also, the plots of the hrf can be decreasing, upside-down bathtub, increasing and J shaped. Therefore, the TLAW distribution is flexible for analyzing lifetime data.

5.1 Some statistical properties for Topp-Leone alpha power Weibull distribution

In this subsection, some basic properties of the TLAW distribution are derived.

5.1.1 The quantile function

The quantile function of TLAW random variable X is given by using (13) or (23), then

$$x_{(\text{TLAW})} = \beta\left\{-\ln\left(1-\frac{\ln\left[(\alpha-1)\left(1-\left(1-U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)\right]}{\ln\alpha}\right)\right\}^{\frac{1}{\lambda}}. \qquad (27)$$

Then, a random sample from TLAW distribution can be generated using (27) where U is random sample from uniform(0,1).

5.1.2 Moments and moment generating function

If $X \sim TLAW(x;\Omega)$, then using (15) the r^{th} moment of X is

$$\mu'_{r(\text{TLAW})} = \frac{\lambda}{\beta^{\lambda}} \sum_{*} D_{k_1,k_2,k_3,k_4} \int_0^{\infty} x^{\lambda+r-1} e^{-\left(\frac{x}{\beta}\right)^{\lambda}} \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right) dx. \quad (28)$$

The moment generating function is obtained by using (15) as follows:

$$M_{x(\text{TLAW})}(t) = \frac{\lambda}{\beta^{\lambda}} \sum_{r=0}^{\infty} \sum_{*} \frac{t^r}{r!} D_{k_1,k_2,k_3,k_4} \int_0^{\infty} x^{\lambda+r-1} e^{-\left(\frac{x}{\beta}\right)^{\lambda}} \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right) dx. \quad (29)$$

5.1.3 Residual life and reversed residual life

The residual lifetime function of TLAW random variable X is given by using (16)

$$R_{\text{TLAW}}(t) = \frac{1-\left[1-\left(1-\frac{\alpha^{1-e^{-(x+t/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}}{1-\left[1-\left(1-\frac{\alpha^{1-e^{-(t/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}}. \quad (30)$$

The reversed residual lifetime function of TLAW random variable X is given by using (17)

$$\bar{R}_{\text{TLAW}}(t) = \frac{1-\left[1-\left(1-\frac{\alpha^{1-e^{-(x-t/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}}{1-\left[1-\left(1-\frac{\alpha^{1-e^{-(t/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}}. \quad (31)$$

5.1.4 Order statistics

From (18), the pdf of the r^{th} order statistic of the TLAW distribution is given by

$$g_{TLAW}\left(x_{(r)}\right)=\frac{2\theta\lambda\ln(\alpha)x_{(r)}^{\lambda-1}}{\beta^{\lambda}(\alpha-1)\boldsymbol{B}(r,n-r+1)}e^{-\left(\frac{x_{(r)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{(r)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(x_{(r)}/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)$$

$$\times\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(x_{(r)}/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{i\theta+r\theta-1},\quad x_{(r)}>0, \tag{32}$$

where $\boldsymbol{B}(.,.)$ denotes the beta function.

5.2 Maximum likelihood estimation

Let $X_1, X_2, \dots, X_n$ be a simple random sampling of size n from the TLAW distribution, then, from the pdf in (24), the likelihood function is

$$L(\Omega;\underline{x})=\left(\frac{2\theta\lambda\ln\alpha}{\beta(\alpha-1)}\right)^{n}\alpha^{\sum_{i=1}^{n}\left(1-e^{-\left(\frac{x_i}{\beta}\right)^{\lambda}}\right)}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\lambda}}\prod_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\lambda-1}$$

$$\times\prod_{i=1}^{n}\left(1-\frac{\alpha^{1-e^{-\left(x_i/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)\prod_{i=1}^{n}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(x_i/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{\theta-1}, \tag{33}$$

where $\Omega=(\lambda,\beta,\alpha,\theta)$.

The log likelihood function is

$$\ell=n\ln\left(\frac{2\theta\lambda\ln\alpha}{\beta(\alpha-1)}\right)+(\lambda-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)+\ln\alpha\sum_{i=1}^{n}\left(1-e^{-\left(\frac{x_i}{\beta}\right)^{\lambda}}\right)$$

$$-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\lambda}+\sum_{i=1}^{n}\ln\left(1-\frac{\alpha^{1-e^{-\left(x_i/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)$$

$$+(\theta-1)\sum_{i=1}^{n}\ln\left(1-\left(1-\frac{\alpha^{1-e^{-\left(x_i/\beta\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right). \tag{34}$$

The *maximum likelihood* (ML) estimators of $\lambda, \beta, \alpha, \text{and}\ \theta$ can be obtained by differentiating (34) with respect to $\lambda, \beta, \alpha, \text{and}\ \theta$ then, equating the results to zeros as follows:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right) + \ln \hat{\alpha} \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}$$

$$- \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}} - \ln \hat{\alpha} \sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\hat{\beta}}\right)\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}} \alpha^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}}{\hat{\alpha} - \hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}}$$

$$+ \frac{2(\hat{\theta}-1)\ln \hat{\alpha}}{\hat{\alpha}-1} \sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\hat{\beta}}\right)\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}} \left(1 - \frac{\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}} - 1}{\hat{\alpha}-1}\right)}{1 - \left(1 - \frac{\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}} - 1}{\hat{\alpha}-1}\right)^2}$$

$$= 0, \tag{35}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{-n}{\hat{\beta}} - \frac{n(\hat{\lambda}-1)}{\hat{\beta}} - \frac{\hat{\lambda} \ln \hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}} \sum_{i=1}^{n} x_i^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}} + \frac{\hat{\lambda}}{\hat{\beta}^{\hat{\lambda}+1}} \sum_{i=1}^{n} x_i^{\hat{\lambda}}$$

$$+ \frac{\hat{\lambda} \ln \hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}} \sum_{i=1}^{n} \frac{x_i^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}}{\hat{\alpha} - \hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}}$$

$$- \frac{2\hat{\lambda}(\hat{\theta}-1)\ln \hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}(\hat{\alpha}-1)} \sum_{i=1}^{n} \frac{x_i^{\hat{\lambda}} e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}} \left(1 - \frac{\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}} - 1}{\hat{\alpha}-1}\right)}{1 - \left(1 - \frac{\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}} - 1}{\hat{\alpha}-1}\right)^2}$$

$$= 0, \tag{36}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\hat{\alpha}\ln\hat{\alpha}} - \frac{n}{\hat{\alpha}-1} + \frac{1}{\hat{\alpha}}\sum_{i=1}^{n} 1 - e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}$$

$$- \frac{1}{\hat{\alpha}-1}\sum_{i=1}^{n} \frac{(\hat{\alpha}-1)\hat{\alpha}^{-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}\left(1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}\right) - \left(\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}}-1\right)}{\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}}$$

$$+ \frac{2(\hat{\theta}-1)}{(\hat{\alpha}-1)}\sum_{i=1}^{n} \frac{\left(\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}\right)\left[(\hat{\alpha}-1)\hat{\alpha}^{-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}\left(1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}\right) - \left(\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}}-1\right)\right]}{(\hat{\alpha}-1)^2 - \left(\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\lambda}}}}\right)^2}$$

$$= 0, \tag{37}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \ln\left(1 - \left(1 - \frac{\hat{\alpha}^{1-e^{-(x_i/\hat{\beta})^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2\right) = 0. \tag{38}$$

It is to be noted that the likelihood equations cannot be solved explicitly, so, the ML estimates of $\lambda, \beta, \alpha,$ and θ can be obtained numerically.

6. Simulation Study

In this section, a simulation study is carried out using Mathematica 9 to evaluate the performance of the ML estimates based on generated data from the TLAW distribution. Evaluating the performance of the estimates is considered through the *relative absolute bias* (RAB) and *estimated risk* (ER), where

$$RAB = \frac{Absolute(estimate - true\ value)}{true\ value}, \tag{39}$$

and

$$ER = \frac{1}{1000}\sum_{i=1}^{1000}(estimate - true\ value)^2\ . \tag{40}$$

The ML averages of the estimates, RAB, and ER for the TLAW distribution are displayed in Tables 2 and 3. The population parameter values used in this simulation study are (λ=4, β=1.5, α=2.5, θ=1.5) and (λ=3, β=2, α=3.5, θ=3), where the number of replications is N=1000 and the samples of size, n=30, 50, 100, and 150.

Table 2: ML averages, relative absolute biases, estimated risks and 95% confidence intervals of the parameters for TLAW distribution (N=1000, λ=4, β=1.5, α=2.5, θ=1.5)

n	Parameter	Average	RAB	ER	LI	UI	Length
30	λ	2.0769	0.4808	3.6987	2.0385	2.1153	0.0768
	β	1.5846	0.0563	0.0076	1.5427	1.6265	0.0839
	α	2.7937	0.1175	0.0878	2.7152	2.8721	0.1568
	θ	0.7532	0.4978	0.5577	0.7360	0.7705	0.0345
50	λ	2.0786	0.4804	3.6920	2.0512	2.1060	0.0548
	β	1.5840	0.0560	0.0073	1.5515	1.6164	0.0649
	α	2.7886	0.1154	0.0843	2.7265	2.8508	0.1243
	θ	0.7535	0.4976	0.5571	0.7407	0.7665	0.0258
100	λ	2.0788	0.4802	3.6907	2.0614	2.0964	0.0350
	β	1.5830	0.0553	0.0070	1.5595	1.6064	0.0469
	α	2.7850	0.1140	0.0817	2.7421	2.8280	0.0859
	θ	0.7539	0.4973	0.5566	0.7451	0.7628	0.0177
150	λ	2.0798	0.4800	3.6871	2.0648	2.0948	0.0300
	β	1.5833	0.0555	0.0070	1.5640	1.6027	0.0386
	α	2.7850	0.1140	0.0815	2.7502	2.8198	0.0696
	θ	0.7539	0.4973	0.5566	0.7462	0.7616	0.0154

Table 3: ML averages, relative absolute biases, estimated risks, and 95% confidence intervals of the parameters for TLAW distribution (N=1000, λ=3, β=2, α=3.5, θ=3)

n	Parameter	Average	RAB	ER	LI	UI	Length
30	λ	1.2054	0.5981	3.2204	1.1910	1.2198	0.0288
	β	1.8318	0.0840	0.0284	1.8048	1.8588	0.0540
	α	3.7224	0.0635	0.0494	3.7153	3.7295	0.0142
	θ	1.5922	0.4692	1.9829	1.5220	1.6625	0.1405
50	λ	1.2063	0.5978	3.2170	1.1955	1.2173	0.0218
	β	1.8311	0.0844	0.0286	1.8093	1.8529	0.0436
	α	3.7218	0.0633	0.0492	3.7169	3.7269	0.0100
	θ	1.5896	0.4701	1.9898	1.5335	1.6457	0.1122
100	λ	1.2064	0.5978	3.2168	1.1986	1.2142	0.0156
	β	1.8316	0.0841	0.0284	1.8165	1.8468	0.0303
	α	3.7211	0.0631	0.0489	3.7177	3.7246	0.0069
	θ	1.5909	0.4696	1.9858	1.5514	1.6304	0.0789
150	λ	1.2064	0.5978	3.2167	1.2000	1.2129	0.0129
	β	1.8317	0.0841	0.0283	1.8191	1.8442	0.0251
	α	3.7209	0.0631	0.0488	3.7182	3.7237	0.0055
	θ	1.5912	0.4695	1.9847	1.5588	1.6237	0.0649

From Tables 2 and 3, one can observe that the ERs of the ML averages of the estimates for the parameters α, β, λ, and θ decrease when the sample size n increases. Also, the lengths of the confidence interval become narrower as the sample size increases.

7. Application

In this section, a real data set of COVID-19 is used to illustrate the applicability of the TLAW distribution and compared with the other lifetime models including *Topp-Leone exponential* (TL-Ex), *alpha power Weibull* (APW), and Weibull distributions. The distribution functions of the competitive models are as follows:

1. The TL-Ex distribution is introduced by Al-Shomrani *et al.* (2016)

$$G_{TL-Ex}(x) = \left(1 - \left(e^{-\frac{x}{\beta}}\right)^2\right)^{\theta}, x > 0;\ \theta, \beta > 0.$$

2. The APW distribution is proposed by Nassar *et al.* (2017)

$$G_{\mathrm{APW}}(x) = \frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}} - 1}{\alpha - 1}, \quad x > 0; \alpha, \lambda, \beta > 0,\ \alpha \neq 1.$$

3. The cdf of the Weibull distribution is given in (21).

To verify which distribution fits better to the real data set, some means for model selection are obtained such as the *Akaike information criterion* (AIC), *consistent Akaike information criterion* (CAIC), *Bayesian information criterion* (BIC), and *Kolmogorov-Smirnov* (KS) statistic to compare the proposed model and other competitive models. The best model corresponds to the lowest values of AIC, CAIC, BIC and KS statistic, which indicates better fit to the data. Figures 2-4 display the PP-plot, QQ-plot, and fitted pdf of the TLAW distribution respectively. Further, Table 4 presents the estimates of the parameters and *standard errors* (SE). The values for AIC, CAIC, BIC, and KS statistic are given in Table 5.

The data given by Mubarak and Almetwally (2021) represents drought mortality rate of COVID-19 data belongs to the United Kingdom of 76 days, from 15 April to 30 June 2020. The data are given as follows:

0.0587	0.0863	0.1165	0.1247	0.1277	0.1303	0.1652	0.2079
0.2395	0.2751	0.2845	0.2992	0.3188	0.3317	0.3446	0.3553
0.3622	0.3926	0.3926	0.4110	0.4633	0.4690	0.4954	0.5139
0.5696	0.5837	0.6197	0.6365	0.7096	0.7193	0.7444	0.8590
1.0438	1.0602	1.1305	1.1468	1.1533	1.2260	1.2707	1.3423
1.4149	1.5709	1.6017	1.6083	1.6324	1.6998	1.8164	1.8392
1.8721	1.9844	2.1360	2.3987	2.4153	2.5225	2.7087	2.7946
3.3609	3.3715	3.7840	3.9042	4.1969	4.3451	4.4627	4.6477
5.3664	5.4500	5.7522	6.4241	7.0657	7.4456	8.2307	9.6315
10.1870	11.1429	11.2019	11.4584				

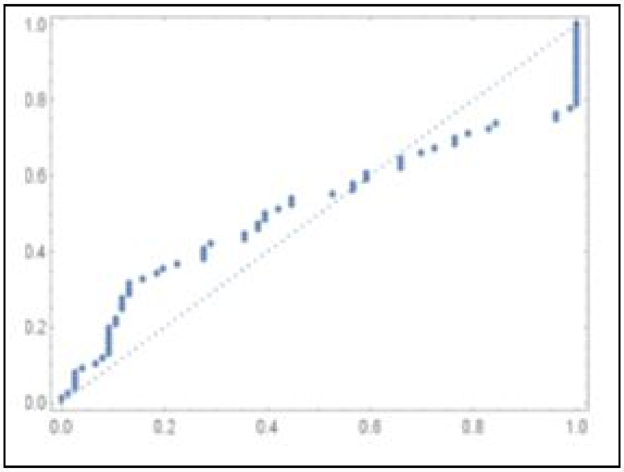

Figure 2: PP plots of the TLAW distribution for the data set

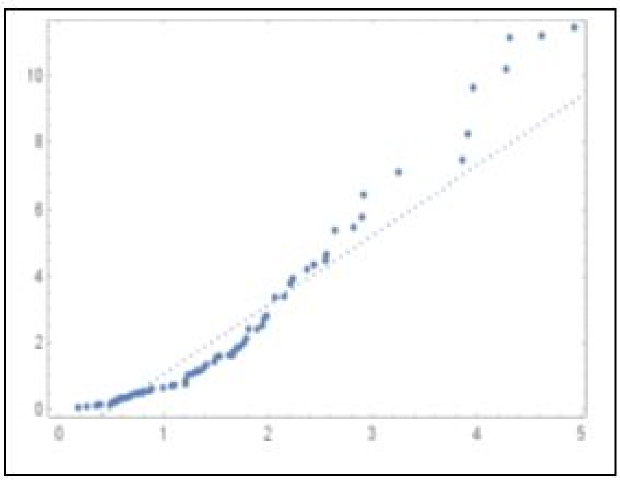

Figure 3: QQ plots of the TLAW distribution for the data set

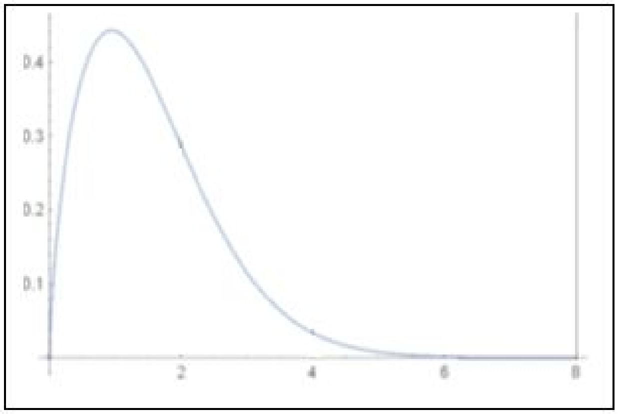

Figure 4: Fitted pdf of the TLAW distribution for the data set

Table 4: ML estimates of the parameters and SE of TLAW distribution for the data set

Parameters	ML estimate	SE
λ	1.2383	0.0208
β	2.9587	0.0462
α	0.7889	0.0204
θ	0.3244	0.0051

Table 5: Criteria for comparison of the fitted models for the data set

Model	KS	*p*-value	AIC	BIC	CAIC
TLAW	0.171	0.217	292.9	302.2	293.4
TL-Ex	0.184	0.152	304.2	308.8	304.3
APW	0.211	0.069	306.4	313.4	306.7
Weibull	0.197	0.103	305.2	309.8	305.3

It can be seen from Tables 4 and 5 that the TLAW distribution provides a better fit than the other distributions regarding the values of AIC, CAIC, BIC, and KS statistic.

8. Conclusion

In this paper, a general class of distributions called TLAP family is presented. Some statistical properties of the general family are studied. A special sub-model is considered namely, TLAW distribution. Estimation of the parameters for TLAW distribution using the ML method is discussed, and a simulation study is carried out. A real data set is applied, and some certain accuracy measures are evaluated. These measures ensure that the TLAW distribution provides a better fit to the real data set than the TL-Ex, APW, and Weibull distributions.

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