Proposed Double Sigmoidal Growth Curves: Modeling and Estimation

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Abstract

Sigmoidal growth curves are a useful tool for modeling experimental growth data when growth proceeds sigmoidally over time. When the changes in response have a double sigmoid growth pattern, it is convenient to employ a double sigmoid growth model to be able to describe the data well. In this paper, new double sigmoidal growth curves are presented based on the Burr Type XII distribution. In addition, for modeling the proposed curves, the procedure of summation of two single sigmoidal growth curves is considered. The proposed models namely the double Burr Type XII–logistic sigmoid growth model and the modified double Burr Type XII–logistic sigmoid growth model. Furthermore, to estimate the parameters of the proposed models, the non-linear least squares and the maximum likelihood methods are used. Moreover, a simulation study and an application are carried out to examine the performance of the proposed models compared to some classical double sigmoid growth models. The results indicate that the proposed model, modified double Burr Type XII–logistic sigmoid growth model is superior to the other existing models.

Keywords**:** *Sigmoidal growth curve, Double sigmoid growth model, Double Burr Type XII–logistic model, Modified double Burr Type XII–logistic model, Non-linear least squares.*

1. Introduction

The sigmoid (S-shaped) curves have been employed in many studies of growth analyses in various fields such as physics, biology, economics, and medicine. The S-curve begins with an exponential growth, slow down when saturation occurs, and completes at maturity. When a single-phase behavior of growth is along the entire path, one of the single sigmoid growth models such as the Brody, logistic, Weibull, and the Burr Type XII sigmoid growth models can be used for describing the growth data. While, when the double-phasic behavior of a growth path occurs because of oscillatory behavior of growth or by a combination of two single phases of growth rate, one of the double sigmoid growth models is preferred such as the double logistic sigmoid growth model by Carrillo and González (2002), and the modified double logistic sigmoid growth model by Fernandes *et al.* (2017). For more details on the sigmoid growth models, one can refer to Tsoularis and Wallace (2002), Seber and Wild (2003), Fernandes *et al.* (2017), Cao *et al.* (2019), Ukalska and Jastrzebowski (2019), and Shen (2020).

Since many growth data have double sigmoidal growth pattern, several studies have been used and proposed double growth models for analyzing the growth data such as Hau *et al*. (1993) used some mathematical functions as double logistic, double Gompertz, monomolecular-logistic, and monomolecular-Gompertz to describe disease progress curves of double sigmoid pattern from epidemics of sugarcane smut, Carrillo and González (2002) analyzed the growth of electricity consumption in the United States by double logistic growth curves, Fernandes *et al*. (2017) used the double logistic and double Gompertz growth models for analyzing the growth pattern of coffee berries, Letchov and Roychev (2017) analyzed the growth kinetics of grape berry by the double logistic sigmoid growth model, Tello and Forneck (2018) described the development of grapevine bunch compactness by a double sigmoid model, El Aferni *et al*. (2021) applied the double sigmoid Boltzmann model to study the COVID-19 spread in fifteen different countries, and Pal and Mitra (2021) analyzed the number of cumulative cases of COVID-19 in Iceland by the double sigmoid Boltzmann model.

The purpose of this paper is to introduce proposed double sigmoidal growth curves based on the Burr Type XII distribution for describing various phenomena that have double sigmoid growth patterns. Also, modeling the new proposed curves is aimed. The rest of this paper is constructed as follows. In Section 2, the proposed curves of double sigmoidal growth and their modeling are presented. In Section 3, the parameters of the proposed models are estimated by the *non-linear least squares* (NLS) and the *maximum likelihood* (ML) estimation methods. In Section 4, a Monte Carlo simulation is carried out to investigate the performance of the proposed models against some existing models of double sigmoidal growth. In Section 5, an application using confirmed new cases of COVID-19 in Egypt in 2020 is provided. At last, the conclusions are shown in Section 6.

2. The Proposed Curves of Double Sigmoidal Growth and their Modeling

There are different procedures for modeling the single sigmoidal curve to the single sigmoid growth models; one important of these procedure formulas is based on the *cumulative distribution function* (CDF) as proposed by Seber and Wild (2003). The general formula of sigmoidal curve based on the distribution function can be defined as follows:

$$f(x) = \beta + (\alpha - \beta)F\big(k(x-\gamma)\big), \tag{1}$$

where x is the independent variable, γ is the inflection point, α is the maximum value of the dependent variable in the data, $\alpha > 0, \beta$ is the minimum value of the response variable, $F(.)$ is the CDF of a continuous random variable, and k is a scale parameter on x , $k > 0$.

By considering the CDF of logistic, and Burr Type XII distributions, the following functions of curves of single sigmoidal growth are given as follows:

$$f_L(x_i, \boldsymbol{\theta}) = \frac{\alpha}{1+e^{[-k(x_i-\gamma)]}} \quad , \boldsymbol{\theta} = (\alpha, k, \gamma)', \tag{2}$$

$$f_B(x_i, \boldsymbol{\theta}) = \big[\beta + (\alpha-\beta)[\,1-(1+(kx_i)^c)^{-r}]\big],\ \boldsymbol{\theta} = (\alpha, \beta, k, c, r)', \tag{3}$$

where $f_L(x_i, \boldsymbol{\theta})$ and $f_B(x_i, \boldsymbol{\theta})$ are the functions of single logistic and Burr Type XII sigmoidal growth curves, $\boldsymbol{\theta}$ is the vector of parameters, and c and r are the shape and scale parameters respectively of the Burr Type XII distribution.

In the presence of the double-phase behavior of the sigmoidal growth curve, the double logistic sigmoidal growth curve is given as follows:

$$f_{DL}(x_i, \boldsymbol{\theta}) = \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}} + \frac{\alpha_2}{1+e^{[-k_2(x_i-\gamma_2)]}}, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)', \tag{4}$$

where α_1 and α_2 are the upper asymptotes in the first and second curves respectively, k_1 and k_2 are the slope factors of the two phases respectively, and γ_1 and γ_2 are the first and second points of inflection with $\gamma_2 > \gamma_1$.

Also, the modified double logistic sigmoidal growth curve can be obtained by the following function:

$$f_{MDL}(x_i, \boldsymbol{\theta}) = \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}} + \frac{\alpha_2-\alpha_1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)', \quad (5)$$

where α_1 and $(\alpha_2 - \alpha_1)$ as two upper asymptotes in the first and second curves respectively.

Consequently, Carrillo and González (2002) introduced the double logistic sigmoid growth model using the summation of two single logistic sigmoidal growth curves in (4) by the following form:

$$y_{i(DL)} = f_{DL}(x_i, \boldsymbol{\theta}) + \varepsilon_i, \quad \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)',$$

$$= \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}} + \frac{\alpha_2}{1+e^{[-k_2(x_i-\gamma_2)]}} + \varepsilon_i, \quad (6)$$

where $y_{i(DL)}$; $i = 1, \ldots, n$ is the response variable in the double logistic sigmoid growth model, x_i is the independent variable, α_1 and α_2 are the upper asymptotes, k_1 and k_2 are related to the initial levels, γ_1 and γ_2 are the first and second points of inflection with $\gamma_2 > \gamma_1$, and ε_i is the random error term which is *independent and identically distributed* $(i.i.d.)$ with $N(0, \sigma^2)$.

Also, the modified double logistic sigmoid growth model was introduced by Fernandes *et al*. (2017) as follows:

$$y_{i(MDL)} = f_{MDL}(x_i, \boldsymbol{\theta}) + \varepsilon_i, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)',$$

$$= \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}} + \frac{\alpha_2-\alpha_1}{1+e^{[-k_2(x_i-\gamma_2)]}} + \varepsilon_i. \quad (7)$$

In this section, proposed two curves of double sigmoidal growth are presented based on two single sigmoidal growth curves. The first new proposed curve called the double Burr Type XII–logistic sigmoidal growth curve in which based on (2) and (3) as follows:

$$f_{DBL}(x_i, \boldsymbol{\theta}) = \left[\beta + (\alpha_1 - \beta)[1 - (1 + (k_1 x_i)^c)^{-r}]\right]$$

$$+ \frac{\alpha_2}{1+e^{[-k_2(x_i-\gamma_2)]}}, \boldsymbol{\theta} = (\alpha_1, \beta, k_1, \alpha_2, k_2, \gamma_2, c, r)'. \quad (8)$$

The second proposed curve is called the modified double Burr Type XII–logistic sigmoidal growth curve. It can be obtained as follows:

$$f_{MDBL}(x_i,\boldsymbol{\theta}) = \left[\beta + (\alpha_1 - \beta)[\,1 - (1 + (k_1 x_i)^c)^{-r}]\right]$$
$$+\frac{\alpha_2-\alpha_1}{1+e^{[-k_2(\,x_i-\gamma_2)]}}\,, \boldsymbol{\theta} = (\alpha_1,\beta,k_1,\alpha_2,k_2,\gamma_2,c,r)'. \quad (9)$$

Consequently, the two proposed models of double sigmoidal growth are given based on (8) and (9) respectively as follows:

The double Burr Type XII–logistic sigmoid growth model:

$$y_{i(DBL)} = f_{DBL}(x_i,\boldsymbol{\theta})+\varepsilon_i,\ \ \boldsymbol{\theta} = (\alpha_1,\beta,k_1,\alpha_2,k_2,\gamma_2,c,r)'$$
$$= \left[\beta + (\alpha_1 - \beta)[\,1 - (1 + (k_1 x_i)^c)^{-r}]\right] + \frac{\alpha_2}{1+e^{[-k_2(\,x_i-\gamma_2)]}} + \varepsilon_i. \quad (10)$$

The modified double Burr Type XII–logistic sigmoid growth model:

$$y_{i(MDBL)} = f_{MDBL}(x_i,\boldsymbol{\theta})+\varepsilon_i, \qquad \boldsymbol{\theta} = (\alpha_1,\beta,k_1,\alpha_2,k_2,\gamma_2,c,r)'$$
$$= \left[\beta + (\alpha_1 - \beta)[\,1 - (1 + (k_1 x_i)^c)^{-r}]\right] + \frac{\alpha_2-\alpha_1}{1+e^{[-k_2(\,x_i-\gamma_2)]}} + \varepsilon_i. \quad (11)$$

3. Estimation of the parameters

In this section, the parameters of the double logistic, modified double logistic, double Burr Type XII–logistic, and modified double Burr Type XII–logistic sigmoid growth models are estimated by the NLE and ML methods.

3.1 Non-linear least squares estimation

Suppose that $\boldsymbol{\theta}$ is the vector of p unknown parameters to be estimated in the proposed models by the NLS method, it is required to minimize the squared residuals:

$$\widehat{\boldsymbol{\theta}}_{\mathrm{NLS}} = \arg\min \textstyle\sum_{\mathrm{i}=1}^{\mathrm{n}}[y_i - f(x_i,\boldsymbol{\theta})]^2. \qquad (12)$$

The first order condition is

$$\textstyle\sum_{i=1}^{n}[y_i - f(x_i,\boldsymbol{\theta})]\frac{\partial f(x,\boldsymbol{\theta})}{\partial \theta_j} = 0\ \ , j = 1,2,\dots,p. \qquad (13)$$

For the double logistic model as in (6), the NLS estimator minimizes:

$$\widehat{\boldsymbol{\theta}}_{\mathrm{NLS}} = \arg\min \sum_{\mathrm{i=1}}^{\mathrm{n}} [y_i - f_{DL}(x_i, \boldsymbol{\theta})]^2, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)'. \quad (14)$$

Then, the derivatives of $f_{DL}(x_i, \boldsymbol{\theta})$ with respect to the parameters are given by

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial \alpha_1} = \frac{1}{1+e^{[-k_1(x_i-\gamma_1)]}}, \quad (15)$$

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial k_1} = \alpha_1(x_i - \gamma_1)\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (16)$$

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial \gamma_1} = -\alpha_1 k_1\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (17)$$

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial \alpha_2} = \frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (18)$$

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial k_2} = \alpha_2(x_i - \gamma_2)\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]}, \quad (19)$$

$$\frac{\partial f_{DL}(x_i,\boldsymbol{\theta})}{\partial \gamma_2} = -\alpha_2 k_2\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]}. \quad (20)$$

By using R programming, *the Levenberg-Marquardt* (L-M) method is implemented to obtain the solution of (15)-(20) numerically.

Also, for the modified double logistic model as in (7), the NLS estimator minimizes:

$$\widehat{\boldsymbol{\theta}}_{\mathrm{NLS}} = \arg\min \sum_{\mathrm{i=1}}^{\mathrm{n}} [y_i - f_{MDL}(x_i, \boldsymbol{\theta})]^2, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)'. \quad (21)$$

Then, the derivatives of $f_{MDL}(x_i, \boldsymbol{\theta})$ with respect to the parameters are given by

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial \alpha_1} = \left[\frac{1}{1+e^{[-k_1(x_i-\gamma_1)]}} - \frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}\right], \quad (22)$$

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial k_1} = \alpha_1(x_i - \gamma_1)\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (23)$$

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial \gamma_1} = -\alpha_1 k_1\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (24)$$

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial \alpha_2} = \frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (25)$$

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial k_2} = (\alpha_2 - \alpha_1)(\, x_i - \gamma_2)\left(1 + e^{[-k_2(\, x_i - \gamma_2)]}\right)^{-2} \; e^{[-k_2(\, x_i - \gamma_2)]}, \quad (26)$$

$$\frac{\partial f_{MDL}(x_i,\boldsymbol{\theta})}{\partial \gamma_2} = -(\alpha_2 - \alpha_1)k_2\left(1 + e^{[-k_2(\, x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(\, x_i - \gamma_2)]}. \quad (27)$$

The solution of (22)-(27) can be obtained numerically by using the L-M iterative method.

For the double Burr Type XII–logistic sigmoid growth model as defined in (10), the NLS estimator minimizes:

$$\widehat{\boldsymbol{\theta}}_{\text{NLS}} = \arg\min \textstyle\sum_{\text{i=1}}^{\text{n}} [y_i - f_{DBL}(x_i, \boldsymbol{\theta})]^2, \boldsymbol{\theta} = (\alpha_1, \beta, k_1, \alpha_2, k_2, \gamma_2, c, r)'. \quad (28)$$

Then, the derivatives of $f_{DBL}(x_i, \boldsymbol{\theta})$ with respect to the parameters are given as follows:

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial \beta} = (1 + (k_1 x_i)^c)^{-r}, \quad (29)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial \alpha_1} = (1 - (1 + (k_1 x_i)^c)^{-r}), \quad (30)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial k_1} = r\, c\, {x_i}^c {k_1}^{c-1}(\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r-1}, \quad (31)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial \alpha_2} = \frac{1}{1 + e^{[-k_2(\, x_i - \gamma_2)]}}, \quad (32)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial k_2} = \alpha_2\left(1 + e^{[-k_2(\, x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(\, x_i - \gamma_2)]}(\, x_i - \gamma_2), \quad (33)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial \gamma_2} = -k_2\alpha_2\left(1 + e^{[-k_2(\, x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(\, x_i - \gamma_2)]}, \quad (34)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial r} = (\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r} \, ln(1 + (k_1 x_i)^c), \quad (35)$$

$$\frac{\partial f_{DBL}(x_i,\boldsymbol{\theta})}{\partial c} = r\,(\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r-1}(k_1 x_i)^c \, ln(k_1 x_i). \quad (36)$$

The solution of (29)-(36) can be obtained numerically by using the L-M iterative method.

For the modified double Burr Type XII–logistic sigmoid growth model as defined in (11), the NLS estimator minimizes:

$$\widehat{\boldsymbol{\theta}}_{\mathrm{NLS}} = \arg\min \sum_{\mathrm{i=1}}^{\mathrm{n}} [y_i - f_{MDBL}(x_i, \boldsymbol{\theta})]^2, \boldsymbol{\theta} = (\alpha_1, \beta, k_1, \alpha_2, k_2, \gamma_2, c, r)'. \quad (37)$$

Then, the derivatives of $f_{MDBL}(x_i, \boldsymbol{\theta})$ with respect to the parameters are given as follows:

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial \beta} = (1 + (k_1 x_i)^c)^{-r}, \quad (38)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial \alpha_1} = (1 - (1 + (k_1 x_i)^c)^{-r}) - \frac{1}{1 + e^{[-k_2(x_i - \gamma_2)]}}, \quad (39)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial k_1} = r\, c\, x_i{}^c k_1{}^{c-1} (\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r-1}, \quad (40)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial \alpha_2} = \frac{1}{1 + e^{[-k_2(x_i - \gamma_2)]}}, \quad (41)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial k_2} = (\alpha_2 - \alpha_1)\left(1 + e^{[-k_2(x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(x_i - \gamma_2)]}(x_i - \gamma_2), \quad (42)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial \gamma_2} = -k_2(\alpha_2 - \alpha_1)\left(1 + e^{[-k_2(x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(x_i - \gamma_2)]}, \quad (43)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial r} = (\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r} \ln(1 + (k_1 x_i)^c), \quad (44)$$

$$\frac{\partial f_{MDBL}(x_i, \boldsymbol{\theta})}{\partial c} = r\,(\alpha_1 - \beta)(1 + (k_1 x_i)^c)^{-r-1}(k_1 x_i)^c \ln(k_1 x_i). \quad (45)$$

Then, using the L-M iterative method, the solution of (38)-(44) can be obtained numerically.

3.2 Maximum likelihood estimation

The *maximum likelihood* (ML) estimation for the parameters of the double logistic, modified double logistic, double Burr Type XII–logistic, and modified double Burr Type XII–logistic sigmoid growth models can be obtained as follows:

For the double logistic sigmoid model as in (6), consider $\boldsymbol{y} = (y_1, \dots, y_n)'$ be n independent random variables with pdf, $f(y_i|\boldsymbol{\theta}, \sigma_\varepsilon^2)$ depending on a vector-valued parameter $(\boldsymbol{\theta})$ and the variance of error, σ_ε^2. Also, the $\varepsilon_i's$ are assumed to be $i.i.d$ with $N(0, \sigma^2)$, then the likelihood function is

$$L = f(\boldsymbol{y}|\boldsymbol{\theta}, \sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-\frac{n}{2}} exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{(y_i-[f_{DL}(x_i,\boldsymbol{\theta})])^2}{\sigma_\varepsilon^2}\right)\right]. \quad (46)$$

The ML estimator of the parameters can be obtained by maximizing the logarithm of the likelihood function (46) denoted by $l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})$ which can be written in the form:

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y}) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{(y_i-[f_{DL}(x_i,\boldsymbol{\theta})])^2}{\sigma_\varepsilon^2}\right). \quad (47)$$

The first partial derivatives of (47) with respect to the parameters are:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)', \quad (48)$$

where:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \alpha_1} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])\frac{1}{1+e^{[-k_1(x_i-\gamma_1)]}}, \quad (49)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial k_1} = -\frac{\alpha_1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])(x_i-\gamma_1)\left(1+e^{[-k_1(x_i-\gamma_1)]}\right)^{-2}e^{[-k_1(x_i-\gamma_1)]}, \quad (50)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \gamma_1} = -\frac{\alpha_1 k_1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])\left(1+e^{[-k_1(x_i-\gamma_1)]}\right)^{-2}e^{[-k_1(x_i-\gamma_1)]}, \quad (51)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \alpha_2} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])\frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (52)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial k_2} = -\frac{\alpha_2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])(x_i-\gamma_2)\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}, \quad (53)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \gamma_2} = -\frac{\alpha_2 k_2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}, \quad (54)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i - [f_{DL}(x_i,\boldsymbol{\theta})])^2. \quad (55)$$

The ML estimators are obtained by setting (49) - (55) equal to zero. The resulting system of non-linear equations can be solved numerically using Nelder–Mead maximization method.

From (7) as the modified double logistic sigmoid model defined, the ε_i's are $i.i.d.\, N(0, \sigma^2)$, then, the likelihood function is given by

$$L = f(\boldsymbol{y}|\boldsymbol{\theta}, \sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2}\, exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])^2}{\sigma_\varepsilon^2}\right)\right]. \quad (56)$$

The log-likelihood function is

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\, \boldsymbol{y}) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])^2}{\sigma_\varepsilon^2}\right). \quad (57)$$

The ML estimator of $\boldsymbol{\theta}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \boldsymbol{\theta} = (\alpha_1, k_1, \gamma_1, \alpha_2, k_2, \gamma_2)', \quad (58)$$

where:

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \alpha_1} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])\left[\frac{1}{1+e^{[-k_1(x_i-\gamma_1)]}} - \frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}\right], \quad (59)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial k_1} = -\frac{\alpha_1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])\,(x_i - \gamma_1)\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (60)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \gamma_1} = -\frac{\alpha_1 k_1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \quad (61)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \alpha_2} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])\frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (62)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial k_2} = -\frac{(\alpha_2-\alpha_1)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])$$

$$\times\,(x_i - \gamma_2)\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}\, e^{[-k_2(x_i-\gamma_2)]}, \quad (63)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2; \boldsymbol{y})}{\partial \gamma_2} = -\frac{(\alpha_2-\alpha_1)k_2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i, \boldsymbol{\theta})])$$

$$\times\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]}, \quad (64)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i - [f_{MDL}(x_i,\boldsymbol{\theta})])^2. \quad (65)$$

The ML estimators are obtained by setting (59) - (65) equal to zero. The resulting system of non-linear equations can be solved numerically using Nelder–Mead maximization method.

For the first proposed model, double Burr Type XII–logistic sigmoid growth model as defined in (10), the $\varepsilon_i's$ are assumed to be $i.i.d$ with $N(0,\sigma^2)$, then, the likelihood function is given as follows:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2}\, exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - f_{DBL}(x_i,\boldsymbol{\theta})\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (66)$$

The log-likelihood function is

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y}) \propto -\frac{n}{2}\log g(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - f_{DBL}(x_i,\boldsymbol{\theta})\right)^2}{\sigma_\varepsilon^2}\right). \quad (67)$$

The ML estimator of $\boldsymbol{\theta}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \boldsymbol{\theta} = (\alpha_1,k_1,\beta,\alpha_2,k_2,\gamma_2,c,r)', \quad (68)$$

where:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \alpha_1} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])\,(1-(1+(k_1x_i)^c)^{-r}), \quad (69)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \beta} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])\,(1+(k_1x_i)^c)^{-r}, \quad (70)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k_1} = -r\,c\,{x_i}^c{k_1}^{c-1}\frac{(\alpha_1-\beta_1)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])\,(1+(k_1x_i)^c)^{-r-1}, \quad (71)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \alpha_2} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])\frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (72)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k_2} = -\frac{\alpha_2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])$$

$$\times\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}(x_i-\gamma_2), \quad (73)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \gamma_2} = \frac{k_2\alpha_2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])$$

$$\times\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]}, \quad (74)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial r} = -\frac{(\alpha_1-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])$$

$$\times (1+(k_1x_i)^c)^{-r}\, ln(1+(k_1x_i)^c), \quad (75)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial c} = -\frac{r\,(\alpha_1-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])$$

$$\times (1+(k_1x_i)^c)^{-r-1}(k_1x_i)^c\, ln(k_1x_i). \quad (76)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i - [f_{DBL}(x_i,\boldsymbol{\theta})])^2. \quad (77)$$

The ML estimators are obtained by setting (69) - (77) equal to zero. The resulting system of non-linear equations can be solved numerically using Nelder–Mead maximization method.

For the second new proposed model, modified double Burr Type XII–logistic sigmoid growth model as defined in (11), the $\varepsilon_i's$ are assumed to be $i.i.d$ with $N(0,\sigma^2)$, then, the likelihood function is as follows:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2}\, exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - f_{MDBL}(x_i,\boldsymbol{\theta})\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (78)$$

The logarithm of the likelihood function (78) is denoted by $l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y})$ which can be written as follows:

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y}) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - f_{MDBL}(x_i,\boldsymbol{\theta})\right)^2}{\sigma_\varepsilon^2}\right). \quad (79)$$

The ML estimator of $\boldsymbol{\theta}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \boldsymbol{\theta} = (\alpha_1, k_1\beta, \alpha_2, k_2, \gamma_2, c, r)', \quad (80)$$

where:

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial \alpha_1} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times\left[(1-(1+(k_1x_i)^c)^{-r}) - \frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}\right], \quad (81)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial \beta} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])\,(1+(k_1x_i)^c)^{-r}, \quad (82)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial k_1} = -r\,c\,{x_i}^c{k_1}^{c-1}\frac{(\alpha_1-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times(1+(k_1x_i)^c)^{-r-1}, \quad (83)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial \alpha_2} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])\frac{1}{1+e^{[-k_2(x_i-\gamma_2)]}}, \quad (84)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial k_2} = -\frac{(\alpha_2-\alpha_1)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}(x_i-\gamma_2), \quad (85)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial \gamma_2} = \frac{k_2(\alpha_2-\alpha_1)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}, \quad (86)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial r} = -\frac{(\alpha_1-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times(1+(k_1x_i)^c)^{-r}\,ln(1+(k_1x_i)^c), \quad (87)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial c} = -\frac{r\,(\alpha_1-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])$$

$$\times(1+(k_1x_i)^c)^{-r-1}(k_1x_i)^c\,ln(k_1x_i), \quad (88)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\mathbf{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i - [f_{MDBL}(x_i,\boldsymbol{\theta})])^2. \quad (89)$$

The ML estimators are obtained by setting (81) - (89) equal to zero. The resulting system of non-linear equations can be solved numerically using Nelder–Mead maximization method.

On the other hand, the initial values of the parameters are needed to obtain the estimators when the iterative methods are used. Then, the starting values of parameters are determined as follows:

The starting values of $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$: The parameters α_{01} and α_{02} of α_1 and α_2 are specified as the maximum value of the dependent variable in two stages of the data.

The starting values of $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$: The parameters k_{01} and k_{02} of k_1 and k_2 are defined as the constant rate at which the response variable approaches its maximum possible value.

The starting values of $\boldsymbol{\gamma_{01}}$ and $\boldsymbol{\gamma_{02}}$: The parameters γ_{01} and γ_{02} are defined as the point of inflection values of the two curves of an independent variable, or it may be the values of the independent variable corresponding to $\frac{\alpha_{01}}{2}$ or $\frac{\alpha_{02}}{2}$ values of the dependent variable.

The starting value of $\boldsymbol{\beta}$: The starting value for β_0 of β was specified by evaluating the model at the start of the growth, and the assumption that β_0 is the minimum of the dependent variable in the data.

The inflection points of double curves

In the nonlinear sigmoid model with inflection points adjusted by the sum of functions, the points are determined in the functions that correspond to each growth phase. Moreover, by following Mischan *et al.* (2015), the inflection points for the proposed curves of double sigmoidal growth, the double logistic, modified double logistic, double Burr Type XII- logistic and modified double Burr Type XII -logistic are derived as follows:

The double logistic curve:

From (4), let $f_{DL}^{(1)}(x_i,\boldsymbol{\theta}) = \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}}$ and $f_{DL}^{(2)}(x_i,\boldsymbol{\theta}) = \frac{\alpha_2}{1+e^{[-k_2(x_i-\gamma_2)]}}$.

For the inflection point of the first curve, the first and second derivatives of $f_{DL}^{(1)}(x_i, \boldsymbol{\theta})$ denoted as $f_{DL}^{(1)'}(x_i, \boldsymbol{\theta})$ and $f_{DL}^{(1)''}(x_i, \boldsymbol{\theta})$ which are given respectively by:

$$f_{DL}^{(1)'}(x_i, \boldsymbol{\theta}) = \alpha_1 k_1 \left(1 + e^{[-k_1(x_i - \gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]}, \tag{90}$$

and

$$f_{DL}^{(1)''}(x_i, \boldsymbol{\theta}) = \alpha_1 k_1^2 \begin{Bmatrix} 2\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-3} e^{[-2k_1(x_i-\gamma_1)]} \\ -\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-2} e^{[-k_1(x_i-\gamma_1)]} \end{Bmatrix}. \tag{91}$$

When $f_{DL}^{(1)''}(x_i, \boldsymbol{\theta}) = 0$, then, $[2\left(1 + e^{[-k_1(x_i-\gamma_1)]}\right)^{-1} e^{[-2k_1(x_i-\gamma_1)]} - 1] = 0$. Hence, $x_i = \gamma_1$. By substituting the new x_i in the first curve, the new value of $\alpha_{\max(1)} = (\frac{1}{2})\, \alpha_1$.

Also, the inflection point in the second curve can be determined as follows:

The first and second derivatives of $f_{DL}^{(2)}(x_i, \boldsymbol{\theta})$ denoted as $f_{DL}^{(2)'}(x_i, \boldsymbol{\theta})$ and $f_{DL}^{(2)''}(x_i, \boldsymbol{\theta})$ which are given as follows:

$$f_{DL}^{(2)'}(x_i, \boldsymbol{\theta}) = \alpha_2 k_2 \left(1 + e^{[-k_2(x_i - \gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]}, \tag{92}$$

and

$$f_{DL}^{(2)''}(x_i, \boldsymbol{\theta}) = \alpha_2 k_2^2 \begin{Bmatrix} 2\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-3} e^{[-2k_2(x_i-\gamma_2)]} \\ -\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-2} e^{[-k_2(x_i-\gamma_2)]} \end{Bmatrix}. \tag{93}$$

When $f_{DL}^{(2)''}(x_i, \boldsymbol{\theta}) = 0$, then, $[2\left(1 + e^{[-k_2(x_i-\gamma_2)]}\right)^{-1} e^{[-2k_2(x_i-\gamma_2)]} - 1] = 0$. Hence, $x_i = \gamma_2$. By substituting the new x_i in the second curve, the new value of $\alpha_{\max(2)} = (\frac{1}{2})\, \alpha_2$.

The modified double logistic curve:

From (5), let $f_{MDL}^{(1)}(x_i, \boldsymbol{\theta}) = \frac{\alpha_1}{1+e^{[-k_1(x_i-\gamma_1)]}}$ and $f_{MDL}^{(2)}(x_i, \boldsymbol{\theta}) = \frac{\alpha_2 - \alpha_1}{1+e^{[-k_2(x_i-\gamma_2)]}}$.

For the inflection point of the first curve, the first and second derivatives of $f_{MDL}^{(1)}(x_i,\boldsymbol{\theta})$ denoted as $f_{MDL}^{(1)'}(x_i,\boldsymbol{\theta})$ and $f_{MDL}^{(1)''}(x_i,\boldsymbol{\theta})$ which are given in (93) and (94) respectively. When$f_{MDL}^{(1)''}(x_i,\boldsymbol{\theta}) = 0$, then, $x_i = \gamma_1$. By substituting the new x_i in the first curve, the new value of $\alpha_{\max(1)} = (\frac{1}{2})\ \alpha_1$.

Also, the inflection point in the second curve can be determined as follows:

The first and second derivatives of $f_{MDL}^{(2)}(x_i,\boldsymbol{\theta})$ denoted as $f_{MDL}^{(2)'}(x_i,\boldsymbol{\theta})$ and $f_{MDL}^{(2)''}(x_i,\boldsymbol{\theta})$ which are given respectively by:

$$f_{MDL}^{(2)'}(x_i,\boldsymbol{\theta}) = (\alpha_2-\alpha_1)k_2\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}, \quad (94)$$

and

$$f_{MDL}^{(2)''}(x_i,\boldsymbol{\theta}) = (\alpha_2-\alpha_1)k_2^2\begin{Bmatrix}2\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-3}e^{[-2k_2(x_i-\gamma_2)]}\\-\left(1+e^{[-k_2(x_i-\gamma_2)]}\right)^{-2}e^{[-k_2(x_i-\gamma_2)]}\end{Bmatrix}. \quad (95)$$

When $f_{DML}^{(2)''}(x_i,\boldsymbol{\theta}) = 0$, then, $x_i = \gamma_2$. By substituting the new x_i in the second curve, the new value of $\alpha_{\max(2)} = (\frac{1}{2})\,(\alpha_2)$, and $\left(\alpha_{\max(2)} - \alpha_{\max(1)}\right) = (\frac{1}{2})\,(\alpha_2-\alpha_1)$.

The double Burr Type XII- logistic curve:

From (8), let $f_{DBL}^{(1)}(x_i,\boldsymbol{\theta}) = \left[\beta + (\alpha_1-\beta)[\,1-(1+(k_1x_i)^c)^{-r}]\right]$ and $f_{DBL}^{(2)}(x_i,\boldsymbol{\theta}) = \frac{\alpha_2}{1+e^{[-k_2(x_i-\gamma_2)]}}$.

For the inflection point of the first curve, the first and second derivatives of $f_{DBL}^{(1)}(x_i,\boldsymbol{\theta})$ denoted as $f_{DBL}^{(1)'}(x_i,\boldsymbol{\theta})$ and $f_{DBL}^{(1)''}(x_i,\boldsymbol{\theta})$ which are given respectively by

$$f_{DBL}^{(1)'}(x_i,\boldsymbol{\theta}) = r\,c\,{k_1}^c(\alpha_1-\beta){x_i}^{c-1}(1+(k_1x_i)^c)^{-r-1}, \quad (96)$$

and

$$f_{DBL}^{(1)''}(x_i,\boldsymbol{\theta}) = r\,c{k_1}^c(\alpha_1-\beta)\begin{Bmatrix}(-r-1){x_i}^{c-1}(c\,{k_1}^c{x_i}^{c-1})(1+(k_1x_i)^c)^{-r-2}\\+(c-1)\ \ {x_i}^{c-2}(1+(k_1x_i)^c)^{-r-1}\end{Bmatrix}.$$

(97)

When $f_{DBL}^{(1)''}(x_i, \boldsymbol{\theta}) = 0$, then, $(1 + (k_1 x_i)^c)^{-r-1} = 0$. Hence, $x_i = \frac{(-1)^{1/c}}{k_1}$. By substituting the new x_i in the first curve, the new value of $\alpha_{\max(1)} = \alpha_1$.

Also, the inflection point in the second curve can be determined as follows:

The first and second derivatives of $f_{DBL}^{(2)}(x_i, \boldsymbol{\theta})$ denoted $f_{DBL}^{(2)'}(x_i, \boldsymbol{\theta})$and $f_{DBL}^{(2)''}(x_i, \boldsymbol{\theta})$ are given as in (94) and (95) respectively.

When $f_{DBL}^{(2)''}(x_i, \boldsymbol{\theta}) = 0$, then, $x_i = \gamma_2$. By substituting the new x_i in the second curve, the new value of $\alpha_{\max(2)} = (\frac{1}{2})\, \alpha_2$.

The modified double Burr Type XII- logistic curve:

From (9), let $f_{MDBL}^{(1)}(x_i, \boldsymbol{\theta}) = \left[\beta + (\alpha_1 - \beta)[\,1 - (1 + (k_1 x_i)^c)^{-r}]\right]$ and $f_{MDBL}^{(2)}(x_i, \boldsymbol{\theta}) = \frac{(\alpha_2 - \alpha_1)}{1 + e^{[-k_2(x_i - \gamma_2)]}}$.

For the inflection point of the first curve, the first and second derivatives of $f_{MDBL}^{(1)}(x_i, \boldsymbol{\theta})$ denoted as $f_{MDBL}^{(1)'}(x_i, \boldsymbol{\theta})$ and $f_{MDBL}^{(1)''}(x_i, \boldsymbol{\theta})$ are given as in (96) and (97) respectively. When $f_{MDBL}^{(1)''}(x_i, \boldsymbol{\theta}) = 0$, then, the new value of $\alpha_{\max(1)} = \alpha_1$.

Also, for the inflection point in the second curve, the first and second derivatives of $f_{MDBL}^{(2)}(x_i, \boldsymbol{\theta})$ denoted as $f_{MDBL}^{(2)'}(x_i, \boldsymbol{\theta})$ and $f_{MDBL}^{(2)''}(x_i, \boldsymbol{\theta})$ are given as in (91) and (92) respectively. When $f_{MDBL}^{(2)''}(x_i, \boldsymbol{\theta}) = 0$, then, $x_i = \gamma_2$. By substituting the new x_i in the second curve, the new value of $\left(\alpha_{\max(2)} - \alpha_{\max(1)}\right) = (\frac{1}{2})\, (\alpha_2 - \alpha_1)$.

4. Simulation Study

In this section, a Monte Carlo simulation is conducted to make a comparison between the performance of the proposed double sigmoid growth models: double Burr Type XII-logistic, and modified double Burr Type XII-logistic sigmoid growth models, against some the existing double sigmoid growth models as double logistic, and modified double logistic sigmoid growth models. The performance of the estimators of the

NLS and ML for the parameters $\theta_j, j = 1,2, \dots, p$ of these models can be evaluated using the *relative absolute bias* (RAB) and *relative mean squared error* (REMSE) as follows:

$$\text{RAB}(\hat{\theta}_j) = \frac{|Mean(\hat{\theta}_j) - \theta_j|}{\theta_j}, \tag{98}$$

$$\text{REMSE}(\hat{\theta}_j) = \frac{MSE(\hat{\theta}_j)}{\theta_j}, \tag{99}$$

where $Mean(\hat{\theta}_j) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j)$ $, N$ is the total number of replications, and $MSE(\hat{\theta}_j) = var(\hat{\theta}_j) + bias^2(\hat{\theta}_j)$.

Also, the asymptotic normality of NLS and ML estimation can be used to compute the *asymptotic* 100 (1- ω)% *confidence intervals*(A.C.I) for θ_j as follows:

$$\hat{\theta}_j \pm Z_{(1-\frac{\omega}{2})} SE(\hat{\theta}_j), \tag{100}$$

where ω is the significance level and $SE(\hat{\theta}_j)$ is the standard error of $\hat{\theta}_j$.

4.1 Simulation design

The following steps are used to compute the NLS, and ML estimates, RAB, REMSE and A.C.I for the existing and the proposed double sigmoid growth models for different sample sizes $n = 200, 300, 400$, and 600. The computation of the simulation study is developed using R program (version 3.6.3). Some functions in R Program such as minpack.lm, bbmle, stats4, and mle2 packages are used to compare the performance of different double sigmoid growth models estimates under the assumption of normal distribution of random errors.

1. For the double Burr Type XII-logistic, and modified double Burr Type XII-logistic sigmoid growth models, generate $X_1 \sim \text{Burr}(c, r)$, where $c = 1.7, r = 2.5$, and $X_2 \sim logistic(0,1)$.
2. For the double logistic and modified double logistic sigmoid growth models, generate $X_1 \sim logistic(0,1)$ and $X_2 \sim logistic(0,1)$.
3. Obtain the explanatory variables X using X_1 and X_2.
4. Generate the values of error, ε_i from the standard normal distribution.
5. Following Caglar *et al.* (2018), simulate intensity noise from uniform distribution and add noise of parameter equal to 0.03.

6. The initial values of the coefficients are chosen as $\alpha_{01} = 5$, $\beta_{01} = 0.7, k_{01} = 5, c_0 = 1.7,\ r_0 = 2.5, \alpha_{02} = 6, \gamma_{01} = 2, \gamma_{02} = 5$, and $k_{02} = 5$.
7. Obtain the response variablesy_i using equations (6), (7), (10), and (11), and add the intensity noise equal of parameter.
8. Obtain the NLS estimates by solving (14) for the double logistic sigmoid growth model, solving (21) for the modified double logistic sigmoid growth model, solving (28) for the double Burr Type XII-logistic sigmoid growth model, and solving (37) for the modified double Burr Type XII-logistic sigmoid growth model.
9. Obtain the ML estimates by solving (48) for the double logistic sigmoid growth model, solving (58) for the modified double logistic sigmoid growth model, solving (68) for the double Burr Type XII-logistic sigmoid growth model, and solving (80) for the modified double Burr Type XII-logistic sigmoid growth model.
10. Compute the RAB, REMSE and $\mathrm{A.C.I}$ for each estimate using (98), (99), and (100) respectively.
11. Repeat the above steps for all double sigmoid models and all sample sizes 5000 times using R program.

The results of the simulation study are summarized in Tables 1-16. These tables give the estimated, RAB, REMSE, and A.C.I for each estimate of the considered double sigmoid growth model. The plots of the fitted different double sigmoid growth models are shown in Figures 1-4.

4.2 Simulation results

The main results of the simulation study indicate that: By comparing the RAB and REMSE of the estimator in all models, the NLS estimation is appropriate than ML estimation which agrees with the theoretical results; in all cases, as n increases, the RAB, REMSE, and the length of A.C.I. decrease. Also, it can be found that the modified double Burr Type XII-logistic sigmoid growth model has the shortest confidence interval than other suggested and existing models by NLS, ML methods in the most sample sizes; and as shown in Figures 1, 2, 3, and 4, it is noticed that, the first and second inflection points are very close of the average estimate values to the inflection points for almost sample sizes in all models by NLS, ML methods.

Table 1: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double Burr Type XII-logistic sigmoid growth model when $n = 200$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7, r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I Lower bound	A.C.I Upper bound	Length
NLS	$\widehat{\alpha_1}$	5.08495	0.01699	0.00144	4.78690	5.17627	0.38937
	$\hat{\beta}$	1.57146	1.24494	1.08492	0.50700	1.66382	0.15682
	$\widehat{k_1}$	9.04417	0.80883	0.27106	8.13816	9.13599	0.99783
	$\hat{c}$	5.58028	2.28252	0.85684	5.08559	6.07498	0.98938
	$\hat{r}$	0.70624	0.71750	0.28702	0.50066	0.72786	0.22720
	$\widehat{\alpha_2}$	5.70533	0.04911	0.01448	5.27288	6.13777	0.86489
	$\widehat{\gamma_2}$	5.06417	0.01283	0.00082	4.87215	5.12611	0.25396
	$\widehat{k_2}$	6.53855	0.30771	0.47343	5.95248	6.85463	0.90215
ML	$\widehat{\alpha_1}$	5.08496	0.01699	0.00145	4.87540	5.29451	0.41911
	$\hat{\beta}$	1.57183	1.24547	1.08583	1.48864	1.65501	0.16637
	$\widehat{k_1}$	9.04458	0.80891	0.27172	8.45267	9.63567	1.18300
	$\hat{c}$	5.58240	2.28376	0.86649	5.08803	6.07744	0.98941
	$\hat{r}$	0.70519	0.71792	0.28853	0.44050	0.72369	0.28319
	$\widehat{\alpha_2}$	5.70533	0.04911	0.01447	5.56238	6.38885	0.82647
	$\widehat{\gamma_2}$	5.06419	0.01289	0.00083	4.89126	5.21678	0.32552
	$\widehat{k_2}$	6.53890	0.30778	0.47364	5.66117	6.61663	0.95546

Table 2: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double Burr Type XII-logistic sigmoid growth model when $n = 300$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7, r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I Lower bound	A.C.I Upper bound	Length
NLS	$\widehat{\alpha_1}$	5.05167	0.01033	0.00053	5.05886	5.31009	0.25123
	$\hat{\beta}$	1.48659	1.12370	0.88389	1.40506	1.50542	0.10036
	$\widehat{k_1}$	8.69674	0.73934	0.27031	8.28913	9.28512	0.99599
	$\hat{c}$	3.27051	0.92382	0.45088	3.06106	3.99607	0.93501
	$\hat{r}$	0.95908	0.61636	0.24976	0.86054	1.08697	0.22643
	$\widehat{\alpha_2}$	6.27828	0.04638	0.01290	6.23346	6.32310	0.08963
	$\widehat{\gamma_2}$	4.97515	0.00496	0.00012	4.92669	5.02362	0.09693
	$\widehat{k_2}$	5.47084	0.09416	0.04433	4.84674	5.62194	0.77520
ML	$\widehat{\alpha_1}$	5.05171	0.01034	0.00054	4.84730	5.25612	0.40882
	$\hat{\beta}$	1.48668	1.12383	0.88410	1.34821	1.49826	0.15005
	$\widehat{k_1}$	8.70131	0.74026	0.27099	8.05805	9.05408	0.99603
	$\hat{c}$	3.27337	0.92551	0.45618	2.37509	3.35833	0.98324
	$\hat{r}$	0.95876	0.61649	0.25016	0.83083	1.08733	0.25650
	$\widehat{\alpha_2}$	6.27828	0.04638	0.01291	6.23346	6.32310	0.25643
	$\widehat{\gamma_2}$	4.97510	0.00498	0.00013	4.89717	5.01188	0.11471
	$\widehat{k_2}$	5.47089	0.09417	0.04434	5.12055	5.90316	0.78261

Table 3: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double Burr Type XII-logistic sigmoid growth model when $n = 400$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7,\ r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.05038	0.01008	0.00050	5.00131	5.20897	0.20765
	$\hat{\beta}$	0.98446	0.40638	0.11560	0.93576	0.98622	0.05046
	$\widehat{k_1}$	7.62931	0.52586	0.23826	6.98924	7.89240	0.90316
	$\hat{c}$	2.41000	0.41765	0.29653	1.56861	2.46923	0.90062
	$\hat{r}$	1.43481	0.42607	0.24538	1.38165	1.58324	0.20159
	$\widehat{\alpha_2}$	5.81612	0.03064	0.00563	5.74061	5.82602	0.08540
	$\widehat{\gamma_2}$	5.01829	0.00365	0.00006	5.01466	5.10184	0.08718
	$\widehat{k_2}$	5.20455	0.04091	0.04089	4.97167	5.43743	0.46575
ML	$\widehat{\alpha_1}$	5.05049	0.01009	0.00051	4.91628	5.28177	0.36549
	$\hat{\beta}$	0.98550	0.40786	0.11644	0.89295	0.99250	0.09955
	$\widehat{k_1}$	7.63088	0.52617	0.26431	7.50809	8.49841	0.99032
	$\hat{c}$	2.41120	0.41835	0.29753	1.72644	2.64144	0.91500
	$\hat{r}$	1.43382	0.42647	0.24546	1.33780	1.546063	0.20826
	$\widehat{\alpha_2}$	5.81607	0.03065	0.00564	5.75894	5.96525	0.20631
	$\widehat{\gamma_2}$	5.01839	0.00367	0.00007	4.92335	5.03731	0.11396
	$\widehat{k_2}$	5.20493	0.04098	0.04099	5.14955	5.71525	0.56570

Table 4: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double Burr Type XII-logistic sigmoid growth model when $n = 600$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7,\ r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.01951	0.00390	0.00007	4.97202	5.02040	0.04838
	$\hat{\beta}$	0.71586	0.02265	0.00035	0.70485	0.72084	0.01599
	$\widehat{k_1}$	5.14084	0.02816	0.00396	5.08354	5.16177	0.07823
	$\hat{c}$	1.81061	0.06506	0.00719	1.76887	1.86210	0.09323
	$\hat{r}$	2.40526	0.03789	0.00359	2.40311	2.41705	0.01394
	$\widehat{\alpha_2}$	5.92109	0.01315	0.00103	5.92070	5.92641	0.00571
	$\widehat{\gamma_2}$	5.00792	0.00158	0.00001	5.00275	5.00922	0.00647
	$\widehat{k_2}$	5.08427	0.01685	0.00142	5.00219	5.08921	0.08702
ML	$\widehat{\alpha_1}$	5.03075	0.00615	0.00018	4.98099	5.03810	0.05711
	$\hat{\beta}$	0.72683	0.03832	0.00102	0.71297	0.73683	0.02386
	$\widehat{k_1}$	5.14764	0.02952	0.00435	5.08787	5.16752	0.07965
	$\hat{c}$	1.84829	0.08722	0.01293	1.78754	1.88475	0.09721
	$\hat{r}$	2.39221	0.04311	0.00464	2.38718	2.40321	0.01603
	$\widehat{\alpha_2}$	5.90732	0.01544	0.00143	5.92252	5.92846	0.00594
	$\widehat{\gamma_2}$	5.08675	0.01735	0.00150	5.08161	5.08843	0.00682
	$\widehat{k_2}$	5.08529	0.01705	0.00145	5.00372	5.09295	0.08923

Table 5: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double Burr Type XII-logistic sigmoid growth model when $n = 200$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7, r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.13078	0.02615	0.00342	4.96833	5.19715	0.22882
	$\hat{\beta}$	1.34857	0.92653	0.60093	1.26598	1.43116	0.16517
	$\widehat{k_1}$	6.65116	0.33023	0.54526	6.27683	7.02549	0.74865
	$\hat{c}$	3.95788	1.32816	0.99884	3.20884	4.20628	0.99744
	$\hat{r}$	0.44407	0.82237	0.69073	0.03000	0.44757	0.41757
	$\widehat{\alpha_2}$	6.19555	0.03259	0.00637	5.70634	6.68579	0.97945
	$\widehat{\gamma_2}$	4.75649	0.04870	0.01185	4.68931	4.82368	0.13437
	$\widehat{k_2}$	6.81955	0.36391	0.66215	6.66196	6.97715	0.31518
ML	$\widehat{\alpha_1}$	5.13426	0.02685	0.00360	4.97232	5.29620	0.32388
	$\hat{\beta}$	1.55930	1.22757	1.05486	1.46400	1.65460	0.19060
	$\widehat{k_1}$	7.77702	0.55540	1.54237	7.28583	8.26820	0.98237
	$\hat{c}$	3.57688	1.10404	0.85170	2.82235	3.81925	0.99690
	$\hat{r}$	0.38446	0.84621	0.79019	0.00590	0.79910	0.79320
	$\widehat{\alpha_2}$	6.19606	0.03267	0.00640	5.64179	6.62955	0.98776
	$\widehat{\gamma_2}$	8.29637	0.65927	0.17321	8.14158	8.45115	0.30957
	$\widehat{k_2}$	6.82389	0.36477	0.66531	6.66495	6.98283	0.31788

Table 6: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double Burr Type XII-logistic sigmoid growth model when $n = 300$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7, r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.05992	0.01198	0.00071	5.04677	5.07307	0.02600
	$\hat{\beta}$	1.22486	0.74981	0.39355	1.20169	1.35200	0.15031
	$\widehat{k_1}$	6.03154	0.20630	0.21281	5.98642	6.48700	0.50058
	$\hat{c}$	2.06019	0.21187	0.07631	2.03523	2.08515	0.04991
	$\hat{r}$	1.16934	0.53226	0.70826	1.15239	1.48628	0.33389
	$\widehat{\alpha_2}$	6.06340	0.01056	0.00067	5.87598	6.07311	0.19713
	$\widehat{\gamma_2}$	5.08277	0.01655	0.00137	4.97821	5.09813	0.11992
	$\widehat{k_2}$	4.81663	0.03667	0.00672	4.58751	4.88211	0.29460
ML	$\widehat{\alpha_1}$	5.07523	0.01504	0.00113	5.06019	5.09027	0.03007
	$\hat{\beta}$	1.24501	0.77859	0.42434	1.22000	1.39410	0.17410
	$\widehat{k_1}$	6.03184	0.20636	0.21294	5.96039	6.68579	0.72540
	$\hat{c}$	2.29241	0.34847	0.20644	2.23477	2.35005	0.11527
	$\hat{r}$	1.02457	0.59016	0.77074	0.85514	1.06235	0.55514
	$\widehat{\alpha_2}$	6.06406	0.01067	0.00068	5.89156	6.09774	0.20618
	$\widehat{\gamma_2}$	5.08362	0.01672	0.00139	4.90344	5.14443	0.24099
	$\widehat{k_2}$	5.21995	0.04399	0.00676	5.03411	5.33215	0.29804

Table 7: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double Burr Type XII-logistic sigmoid growth model when $n = 400$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7,\ r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.03186	0.00637	0.00020	5.02647	5.03726	0.01078
	$\hat{\beta}$	0.93869	0.34099	0.08139	0.89064	0.98674	0.09610
	$\widehat{k_1}$	5.11564	0.02312	0.19061	4.81289	5.13580	0.32291
	$\hat{c}$	2.03862	0.19918	0.06744	2.02666	2.05430	0.02764
	$\hat{r}$	1.89275	0.24289	0.14749	1.87747	2.16149	0.28402
	$\widehat{\alpha_2}$	6.04134	0.00689	0.00028	6.01434	6.06945	0.05511
	$\widehat{\gamma_2}$	4.95478	0.00904	0.00040	4.94523	4.96434	0.01910
	$\widehat{k_2}$	4.92619	0.01476	0.00108	4.81496	4.93211	0.11715
ML	$\widehat{\alpha_1}$	5.03484	0.00696	0.00024	5.02903	5.04065	0.01161
	$\hat{\beta}$	0.98321	0.40459	0.11458	0.92880	1.03763	0.10882
	$\widehat{k_1}$	5.39661	0.07932	0.19729	5.00575	5.42827	0.42252
	$\hat{c}$	2.04670	0.20394	0.07071	2.01633	2.10771	0.09138
	$\hat{r}$	3.40011	0.36004	0.32408	3.12665	3.44179	0.31514
	$\widehat{\alpha_2}$	6.04190	0.00698	0.00029	6.01366	6.06902	0.05535
	$\widehat{\gamma_2}$	4.95291	0.00941	0.00044	4.94309	4.96272	0.01962
	$\widehat{k_2}$	4.91808	0.01638	0.00134	4.80396	4.94316	0.13920

Table 8: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double Burr Type XII-logistic sigmoid growth model when $n = 600$ and $\alpha_{01} = 5, \beta_0 = 0.7, k_{01} = 5, c_0 = 1.7,\ r_0 = 2.5, \alpha_{02} = 6, \gamma_{02} = 5$, and $k_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.00712	0.00142	0.00001	5.00601	5.00942	0.00341
	$\hat{\beta}$	0.70114	0.00162	0.00001	0.69704	0.70538	0.00834
	$\widehat{k_1}$	5.06907	0.01381	0.00095	5.06289	5.06947	0.00658
	$\hat{c}$	1.73158	0.01857	0.00058	1.73013	1.73182	0.00169
	$\hat{r}$	2.51172	0.00468	0.00005	2.51139	2.51203	0.00064
	$\widehat{\alpha_2}$	6.00175	0.00029	0.00000	6.00110	6.00182	0.00072
	$\widehat{\gamma_2}$	5.00193	0.00038	0.00000	5.00107	5.00201	0.00094
	$\widehat{k_2}$	5.02711	0.00542	0.00014	5.02092	5.02833	0.00741
ML	$\widehat{\alpha_1}$	5.00836	0.00167	0.00001	0.00619	0.00971	0.00352
	$\hat{\beta}$	0.70225	0.00321	0.00002	0.69453	0.70317	0.00864
	$\widehat{k_1}$	5.07542	0.01508	0.00113	5.07118	5.0784	0.00722
	$\hat{c}$	1.73267	0.01921	0.00062	1.73229	1.73446	0.00217
	$\hat{r}$	2.51264	0.00505	0.00006	2.51222	2.51293	0.00071
	$\widehat{\alpha_2}$	6.00212	0.00035	0.00000	6.00194	6.00269	0.00075
	$\widehat{\gamma_2}$	5.00431	0.00086	0.00000	5.00345	5.00442	0.00097
	$\widehat{k_2}$	5.03927	0.00785	0.00030	5.03218	5.03984	0.00766

Table 9: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double logistic sigmoid growth model when $n = 200$ and $\alpha_{01} = 5,\ k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5,$ and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	4.71323	0.05735	0.01644	4.48854	4.89410	0.40556
	$\widehat{k_1}$	5.72308	0.14461	0.10456	4.95089	5.84190	0.89101
	$\widehat{\gamma_1}$	2.09786	0.04893	0.00978	1.83459	2.16413	0.32954
	$\widehat{\alpha_2}$	6.20521	0.03420	0.00720	5.49512	6.22051	0.72539
	$\widehat{k_2}$	3.07289	0.38542	0.74275	2.41966	3.39101	0.97135
	$\widehat{\gamma_2}$	4.83686	0.03262	0.00532	4.67428	4.88271	0.20843
ML	$\widehat{\alpha_1}$	4.71313	0.05737	0.01645	4.49922	4.99227	0.49305
	$\widehat{k_1}$	5.72323	0.14464	0.10461	4.94471	5.84421	0.89951
	$\widehat{\gamma_1}$	2.09787	0.04894	0.00979	1.81206	2.16625	0.35419
	$\widehat{\alpha_2}$	6.20534	0.03422	0.00722	5.49146	6.22537	0.73391
	$\widehat{k_2}$	3.07408	0.38518	0.74183	2.34764	3.31350	0.96586
	$\widehat{\gamma_2}$	4.83685	0.03263	0.00533	4.68640	4.90114	0.21474

Table 10: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double logistic sigmoid growth model when $n = 300$ and $\alpha_{01} = 5,\ k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5,$ and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.11906	0.02381	0.00283	4.86290	5.12147	0.25857
	$\widehat{k_1}$	5.64779	0.12955	0.08392	5.03523	5.74661	0.71138
	$\widehat{\gamma_1}$	1.95690	0.02154	0.00428	1.93687	2.15228	0.21541
	$\widehat{\alpha_2}$	5.85076	0.02487	0.00371	5.36343	5.88160	0.51817
	$\widehat{k_2}$	4.16975	0.16604	0.13786	3.46453	4.21350	0.74897
	$\widehat{\gamma_2}$	4.90489	0.01902	0.00180	4.79621	4.98241	0.18620
ML	$\widehat{\alpha_1}$	5.11910	0.02382	0.00284	4.85515	5.12144	0.26629
	$\widehat{k_1}$	5.64783	0.12956	0.08393	4.99263	5.72217	0.72954
	$\widehat{\gamma_1}$	1.95689	0.02155	0.00429	1.92171	2.14881	0.22710
	$\widehat{\alpha_2}$	5.85069	0.02488	0.00372	5.41462	5.94273	0.52811
	$\widehat{k_2}$	4.16933	0.16613	0.13800	3.63679	4.38920	0.75241
	$\widehat{\gamma_2}$	4.90488	0.01903	0.00181	4.79873	4.98622	0.18749

Table 11: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double logistic sigmoid growth model when $n = 400$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.10738	0.02147	0.00230	5.05152	5.26157	0.21005
	$\widehat{k_1}$	4.63236	0.07352	0.02703	4.27448	4.77920	0.50472
	$\widehat{\gamma_1}$	1.99714	0.00143	0.00410	1.83245	2.01238	0.17993
	$\widehat{\alpha_2}$	5.93424	0.01095	0.00136	5.79333	5.97426	0.18093
	$\widehat{k_2}$	5.14030	0.02806	0.00393	5.08906	5.14429	0.05523
	$\widehat{\gamma_2}$	4.91673	0.01665	0.00138	4.91204	5.02950	0.11746
ML	$\widehat{\alpha_1}$	5.10744	0.02148	0.00231	5.04304	5.27933	0.23629
	$\widehat{k_1}$	4.63233	0.07353	0.02704	4.26734	4.78941	0.52207
	$\widehat{\gamma_1}$	1.99713	0.00144	0.00411	1.84817	2.03107	0.18290
	$\widehat{\alpha_2}$	5.93419	0.01096	0.00137	5.80699	5.98832	0.18133
	$\widehat{k_2}$	5.14108	0.02821	0.00398	5.10513	5.16257	0.05744
	$\widehat{\gamma_2}$	4.91671	0.01666	0.00139	4.91391	5.03161	0.11770

Table 12: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the double logistic sigmoid growth model when $n = 600$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.04133	0.00826	0.00034	5.01461	5.05438	0.03977
	$\widehat{k_1}$	5.09809	0.01961	0.00192	5.06069	5.11343	0.05274
	$\widehat{\gamma_1}$	1.99842	0.00079	0.00001	1.97828	2.00761	0.02933
	$\widehat{\alpha_2}$	5.98914	0.00181	0.00001	5.93509	5.99045	0.05536
	$\widehat{k_2}$	5.11452	0.02290	0.00262	5.07925	5.12676	0.04751
	$\widehat{\gamma_2}$	4.97395	0.00521	0.00013	4.90586	4.98114	0.07528
ML	$\widehat{\alpha_1}$	5.04422	0.04422	0.00039	5.01535	5.05516	0.03981
	$\widehat{k_1}$	5.09836	0.09836	0.00193	5.08389	5.13871	0.05482
	$\widehat{\gamma_1}$	2.00911	0.00911	0.00004	1.98703	2.01917	0.03214
	$\widehat{\alpha_2}$	6.01582	0.01582	0.00004	5.98105	6.03742	0.05637
	$\widehat{k_2}$	5.13381	0.13381	0.00358	5.09803	5.14582	0.04779
	$\widehat{\gamma_2}$	4.96744	0.00651	0.00021	4.91120	4.98751	0.07631

Table 13: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double logistic sigmoid growth model when $n = 200$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	5.21960	0.04392	0.00964	4.99719	5.39940	0.40221
	$\widehat{k_1}$	3.99500	0.20098	0.20196	3.37716	4.02619	0.64903
	$\widehat{\gamma_1}$	2.04539	0.02269	0.00103	1.74510	2.16840	0.42330
	$\widehat{\alpha_2}$	6.55394	0.09232	0.05114	5.85045	6.56907	0.71862
	$\widehat{k_2}$	1.15964	0.76807	0.94966	0.97154	1.79252	0.82098
	$\widehat{\gamma_2}$	6.36407	0.27281	0.37214	6.19997	6.43673	0.23676
ML	$\widehat{\alpha_1}$	5.21983	0.04396	0.00966	4.98835	5.39962	0.41127
	$\widehat{k_1}$	3.99543	0.20091	0.20182	3.41281	4.04713	0.63432
	$\widehat{\gamma_1}$	2.04543	0.02271	0.00104	1.73141	2.16436	0.43295
	$\widehat{\alpha_2}$	6.55405	0.09234	0.05116	5.85358	6.57241	0.71883
	$\widehat{k_2}$	1.15926	0.76814	0.95024	1.00336	1.82730	0.82394
	$\widehat{\gamma_2}$	6.36409	0.27282	0.37215	6.19974	6.43715	0.23741

Table 14: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double logistic sigmoid growth model when $n = 300$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I		Length
					Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	4.82964	0.03407	0.00580	4.65714	4.88673	0.22959
	$\widehat{k_1}$	5.85103	0.17020	0.14485	5.48897	5.92715	0.43818
	$\widehat{\gamma_1}$	1.97178	0.01410	0.00038	1.70101	1.98251	0.28150
	$\widehat{\alpha_2}$	6.28753	0.04792	0.01377	5.68087	6.30926	0.62839
	$\widehat{k_2}$	1.23799	0.75240	0.83054	0.63712	1.35211	0.71499
	$\widehat{\gamma_2}$	6.15644	0.23128	0.26747	5.98263	6.18746	0.20483
ML	$\widehat{\alpha_1}$	4.82962	0.03408	0.00581	4.65668	4.88721	0.23053
	$\widehat{k_1}$	5.85126	0.17025	0.14492	5.44917	5.89143	0.44226
	$\widehat{\gamma_1}$	1.97177	0.01411	0.00039	1.68616	1.98329	0.29713
	$\widehat{\alpha_2}$	6.28755	0.04793	0.01378	5.75505	6.38390	0.62885
	$\widehat{k_2}$	1.23789	0.75242	0.83055	0.63849	1.35491	0.71642
	$\widehat{\gamma_2}$	6.15646	0.23129	0.26748	5.96950	6.18946	0.21996

Table 15: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double logistic sigmoid growth model when $n = 400$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I Lower bound	A.C.I Upper bound	Length
NLS	$\widehat{\alpha_1}$	5.10286	0.02057	0.00211	4.96009	5.16400	0.20391
	$\widehat{k_1}$	5.84201	0.16840	0.14179	5.60891	5.86301	0.25410
	$\widehat{\gamma_1}$	2.01642	0.00821	0.00013	1.9725	2.18553	0.21303
	$\widehat{\alpha_2}$	6.02173	0.00362	0.00795	5.77432	6.16943	0.39511
	$\widehat{k_2}$	5.51496	0.10299	0.05303	5.39894	5.58573	0.18679
	$\widehat{\gamma_2}$	5.27663	0.05532	0.01530	5.10534	5.29128	0.18594
ML	$\widehat{\alpha_1}$	5.10295	0.02059	0.00212	4.96083	5.16837	0.20754
	$\widehat{k_1}$	5.84204	0.16841	0.14180	5.61710	5.87323	0.25613
	$\widehat{\gamma_1}$	2.01643	0.00822	0.00014	1.97268	2.18604	0.21336
	$\widehat{\alpha_2}$	6.05392	0.00898	0.00542	5.77178	6.16891	0.39713
	$\widehat{k_2}$	5.51914	0.10382	0.05390	5.39768	5.58497	0.18729
	$\widehat{\gamma_2}$	5.27669	0.05533	0.01531	5.11172	5.29789	0.18617

Table 16: The average of the estimated parameter values, RAB, REMSE, and A.C.I of the modified double logistic sigmoid growth model when $n = 600$ and $\alpha_{01} = 5$, $k_{01} = 5, \gamma_{01} = 2, \alpha_{02} = 6, k_{02} = 5$, and $\gamma_{02} = 5$.

Method	Estimator	Average Estimate	RAB	REMSE	A.C.I Lower bound	A.C.I Upper bound	Length
NLS	$\widehat{\alpha_1}$	5.03901	0.00780	0.00030	5.03443	5.04838	0.01395
	$\widehat{k_1}$	5.07029	0.01405	0.00098	5.04104	5.08333	0.04229
	$\widehat{\gamma_1}$	1.99972	0.00014	0.00001	1.99397	2.02031	0.02634
	$\widehat{\alpha_2}$	5.99416	0.00097	0.00001	5.95849	5.99763	0.03914
	$\widehat{k_2}$	5.05742	0.01148	0.00065	4.99624	5.05917	0.06293
	$\widehat{\gamma_2}$	4.98215	0.00357	0.00006	4.93302	4.99216	0.05914
ML	$\widehat{\alpha_1}$	5.04377	0.00875	0.00038	5.0299	5.04653	0.01663
	$\widehat{k_1}$	5.08901	0.01780	0.00158	5.03674	5.09045	0.05371
	$\widehat{\gamma_1}$	1.99895	0.00052	0.00001	1.99698	2.02582	0.02884
	$\widehat{\alpha_2}$	5.99687	0.00052	0.00001	5.96269	6.00452	0.04183
	$\widehat{k_2}$	5.07325	0.01465	0.00107	5.01266	5.08077	0.06811
	$\widehat{\gamma_2}$	4.95749	0.00850	0.00036	5.03047	5.09328	0.06281

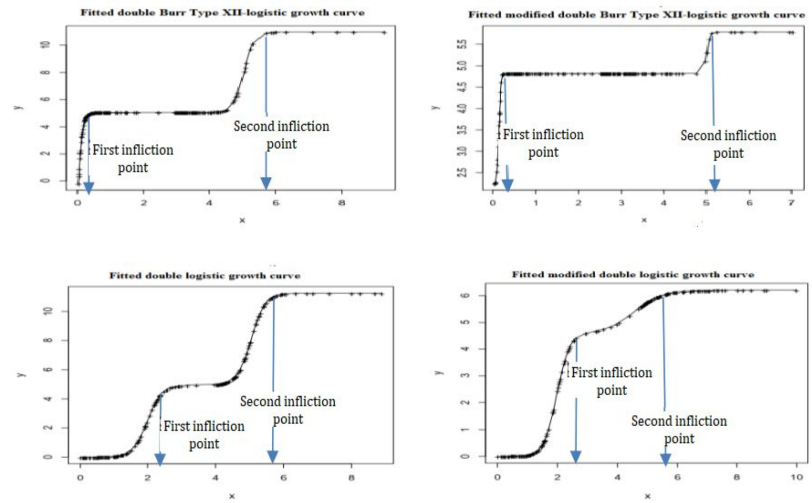

Figure 1: Plots of the fitted double growth curves when $n = 200$.

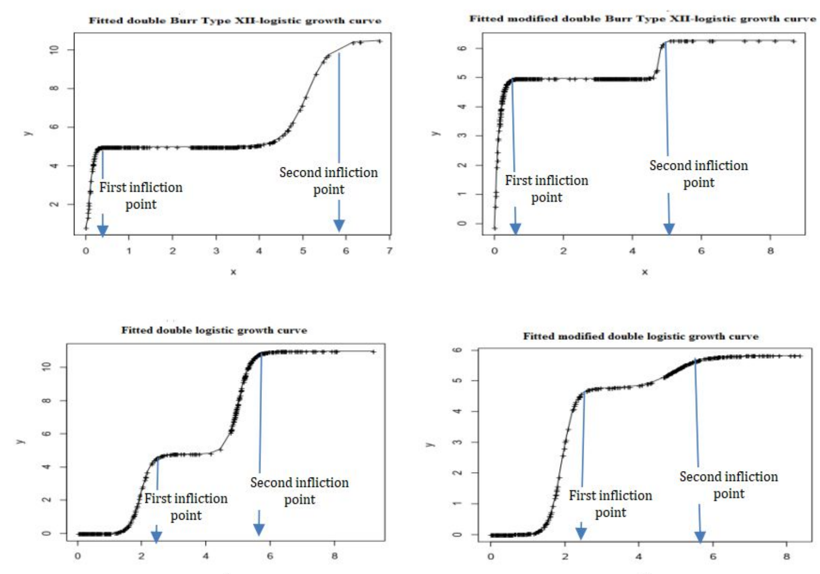

Figure 2: Plots of the fitted double growth curves when $n = 300$.

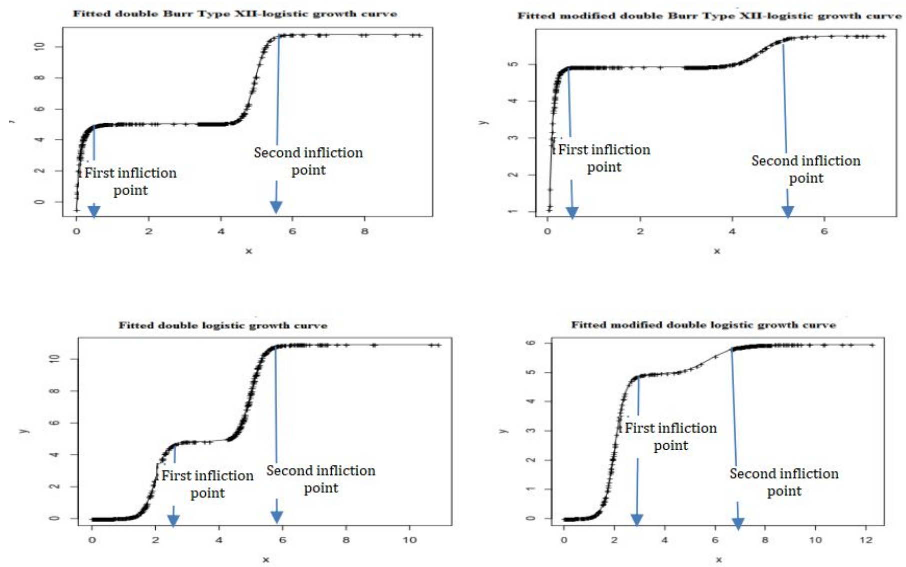

Figure 3: Plots of the fitted double growth curves when $n = 400$.

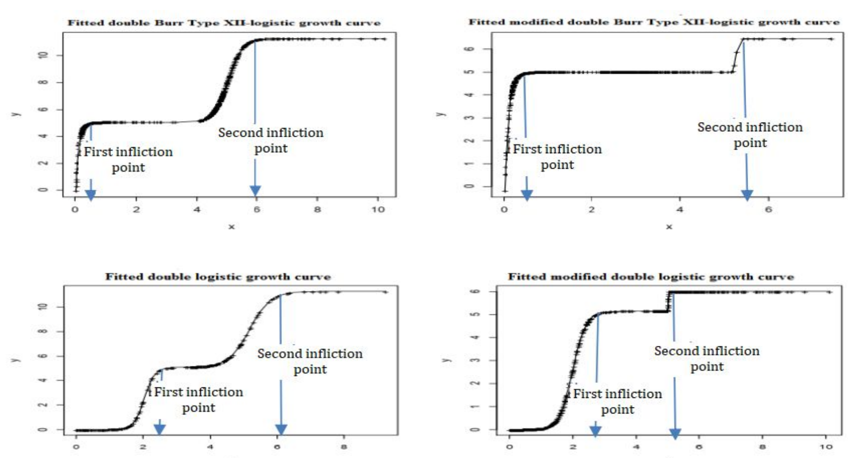

Figure 4: Plots of the fitted double growth curves when $n = 600$.

5. Application

In this section, the COVID-19 data set is analyzed to demonstrate how the proposed models can be used in practice. The data set on the number of daily confirmed new COVID-19 cases in Egypt from March 15, 2020 to May 31, 2020 in the first stage, and the number of daily confirmed new COVID-19 cases in Egypt from June 1, 2020 to September 8, 2020 in the second stage, which are taken from ministry of health and population in Egypt (2020) are used. The data was recorded every day for a period of 178 days (106 days in the first stage, and 72 days in the second stage) as follows:

16	40	30	14	46	29	9	33	39	36
54	39	41	40	33	47	54	69	86	120
85	103	149	128	110	139	95	145	126	125
160	155	168	171	188	112	189	157	169	232
201	227	215	248	260	226	269	358	298	272
348	388	387	393	495	488	436	346	347	338
398	399	491	510	535	720	745	774	783	727
752	702	789	910	1127	1289	1367	1536	1399	1152
1079	1152	1348	1497	1467	1365	1385	1455	1442	1577
1677	1618	1691	1567	1363	1218	1774	1547	1475	1576
1332	1420	1569	1625	1168	1265	1566	1557	1503	1458
1412	1324	1218	969	1057	1025	950	981	923	912
931	929	913	928	703	698	603	627	676	667
668	659	511	479	420	465	409	401	321	238
167	157	112	123	131	141	167	178	174	168
129	145	112	116	139	115	163	161	111	123
89	103	138	141	206	237	223	212	230	212
176	165	145	157	130	151	178	187		

These data are refined by using the inverted variance transformation. The graphical presentation of the relationship between the number of confirmed new cases of COVID-19 as the response variable, and the days as the explanatory variable is shown in Figure 5. The data is characterized by two consecutive stages, one with an increasing stage followed by a decreasing stage.

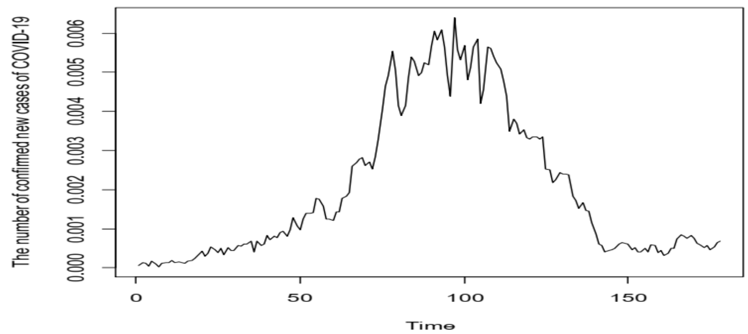

Figure 5: Description of the number of confirmed new cases of COVID-19 over time.

The starting initial values of some initial values are calculated as $a_{01} = 0.00639$, $\beta_{01} = 0.00003$, $a_{02} = 0.00564$, $\gamma_{01} = 81$, $\gamma_{02} = 135$, and some initial values are chosen as $c_0 = 0.2$, $r_0 = 1.9$, $k_{01} = 0.009$, and $k_{02} = 0.013$. The plots of growth curves, double Burr Type XII-logistic, modified double Burr Type XII-logistic, double logistic and modified double logistic using their inflection points are displayed in Figure 6, and the fitted growth curves of each model for the data set are displayed in Figure 7. Also, for comparing between models the parameter estimates, *approximate standard errors* (ASE) and A.C.I at %95 of parameters for each model are summarized in Tables 17-20.

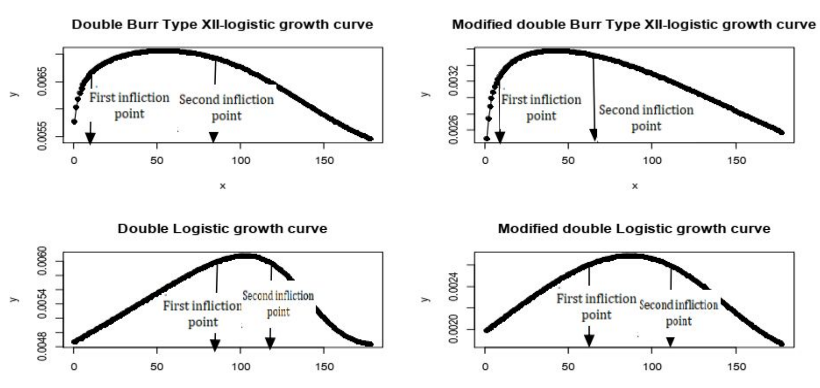

Figure 6: Plots of the growth curves with their respective inflection points.

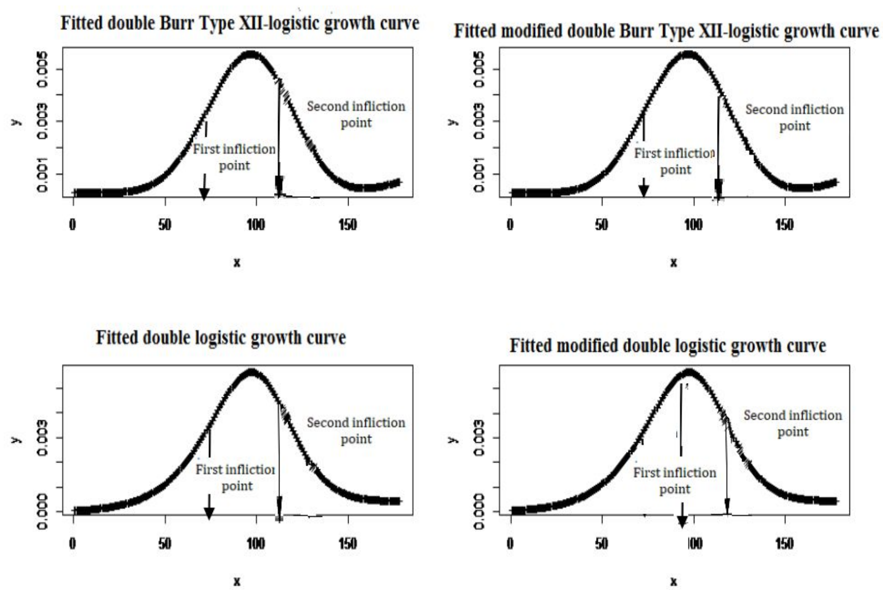

Figure7: Plots of the fitted growth curves.

Table 17: Parameter estimates, ASE, and A.C.I of the parameters for the double Burr Type XII-logistic growth model.

Method	Estimator	Estimates	ASE	A.C.I		Length
				Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	0.00175	0.00242	0	0.00652	0.00652
	$\hat{\beta}$	-0.02532	0.04267	-0.10895	0.05830	0.16725
	$\widehat{k_1}$	0.00881	0.00832	0	0.02514	0.02514
	$\hat{c}$	4.26307	1.00593	2.29147	6.23466	3.94319
	$\hat{r}$	1.41347	3.81389	0	8.88857	8.88857
	$\widehat{\alpha_2}$	0.02566	0.04284	0	0.10963	0.10963
	$\widehat{\gamma_2}$	113.28876	4.59184	104.28890	122.28862	17.99972
	$\widehat{k_2}$	0.06347	0.01632	0	0.631470	0.63147
ML	$\widehat{\alpha_1}$	0.00521	1.94200	0	3.81192	3.81192
	$\hat{\beta}$	-0.01132	2.13100	-4.18780	4.16515	8.35295
	$\widehat{k_1}$	0.01454	1.23600	0	2.43658	2.43658
	$\hat{c}$	4.96700	8.14500	0	16.01440	16.01440
	$\hat{r}$	0.26950	6.47800	0	12.72345	12.72345
	$\widehat{\alpha_2}$	0.01160	2.13500	0	4.19619	4.19619
	$\widehat{\gamma_2}$	121.50000	19.65000	0	139.72870	139.72870
	$\widehat{k_2}$	0.08107	1.19800	0	2.40680	2.40680

Table 18: Parameter estimates, ASE, and A.C.I of the parameters for the modified double Burr Type XII-logistic growth model.

Method	Estimator	Estimates	ASE	A.C.I		Length
				Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	0.02091	0.02132	0	0.06270	0.06270
	$\hat{\beta}$	0.00033	0.00010	0.00012	0.00054	0.00042
	$\widehat{k_1}$	0.01125	0.00633	0	0.02366	0.02366
	$\hat{c}$	4.70500	1.41953	1.92250	7.48699	5.56449
	$\hat{r}$	0.67940	1.10200	0	2.83928	2.83928
	$\widehat{\alpha_2}$	0.00262	0.00197	0	0.00649	0.00649
	$\widehat{\gamma_2}$	114.90000	8.09772	98.98157	130.72410	31.74253
	$\widehat{k_2}$	0.06642	0.02349	0.02038	0.11246	0.09208
ML	$\widehat{\alpha_1}$	0.00611	0.94500	0	1.85825	1.85825
	$\hat{\beta}$	0.00038	0.17020	-0.33329	0.33405	0.66734
	$\widehat{k_1}$	0.01050	3.89000	0	7.63433	7.63433
	$\hat{c}$	6.05100	1.58700	0	3.11599	3.11599
	$\hat{r}$	4.35400	8.49100	0	1.66454	1.66454
	$\widehat{\alpha_2}$	0.00044	0.20940	0	0.41082	0.41082
	$\widehat{\gamma_2}$	121.400	38.05000	0	135.79530	135.7953
	$\widehat{k_2}$	0.10950	2.69900	0	5.30046	5.30046

Table 19: Parameter estimates, ASE, and A.C.I of the parameters for the double logistic growth model.

Method	Estimator	Estimate	ASE	A.C.I		Length
				Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	0.01627	0.01695	0	0.04951	0.04951
	$\widehat{\gamma_1}$	83.95867	9.20024	65.92652	101.99081	36.06429
	$\widehat{k_1}$	0.06479	0.00460	0	0.06575	0.06575
	$\widehat{\alpha_2}$	0.01606	0.01699	0	0.01723	0.01723
	$\widehat{\gamma_2}$	104.92096	12.73836	79.95423	129.88770	49.93347
	$\widehat{k_2}$	0.07232	0.01077	0	0.08121	0.08121
ML	$\widehat{\alpha_1}$	0.00754	2.44900	0	4.80780	4.80780
	$\widehat{\gamma_1}$	74.5600	8.62800	0	86.06706	86.06706
	$\widehat{k_1}$	0.07419	1.81200	0	5.58224	5.58224
	$\widehat{\alpha_2}$	0.00711	2.50500	0	4.90250	4.90250
	$\widehat{\gamma_2}$	118.30000	52.73000	0	136.10453	136.10453
	$\widehat{k_2}$	0.09888	2.45600	0	8.23965	.239658

Table 20: Parameter estimates, ASE, and A.C.I of the parameters for the modified double logistic growth model.

Method	Estimator	Estimate	ASE	A.C.I		Length
				Lower bound	Upper bound	
NLS	$\widehat{\alpha_1}$	0.04250	0.48256	0	0.98836	0.98836
	$\widehat{\gamma_1}$	100.11280	5.49653	0	107.8428	107.8428
	$\widehat{k_1}$	0.06108	0.02810	0	0.11617	0.11617
	$\widehat{\alpha_2}$	0.00053	0.00032	0	0.00116	0.00116
	$\widehat{\gamma_2}$	107.6808	31.94840	45.06294	170.2987	125.23576
	$\widehat{k_2}$	0.06945	0.06308	0	0.19310	0.19310
ML	$\widehat{\alpha_1}$	0.00722	1.48300	0	2.91316	2.91316
	$\widehat{\gamma_1}$	73.32000	5.41500	0	106.87140	106.87140
	$\widehat{k_1}$	0.07687	1.42500	0	2.79978	2.79978
	$\widehat{\alpha_2}$	0.00043	0.21200	0	0.41592	0.41592
	$\widehat{\gamma_2}$	118.90000	38.06000	0	157.87960	157.87960
	$\widehat{k_2}$	0.10120	2.18900	0	4.30054	4.30054

From Tables 17-20, by comparing the NLS, ML methods for each model, the results indicate that the ASE and A.C.I in the NLS method are better than the ML method.

For choosing the best model in describing the data, the following criteria are used: the coefficient of determination, R^2, *mean squared error* (MSE), *root mean squared error* (RMSE) and *model efficiency* (ME) are shown in Table 21 according to the following formulas:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2+\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}, \tag{101}$$

$$\text{MSE} = \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n}, \tag{102}$$

$$\text{RMSE} = \sqrt{\text{MSE}}, \tag{103}$$

$$\text{ME} = 1 - \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2}, \tag{104}$$

where n is the sample size, y_i , $\hat{y}_i$ are the observed and predicted values respectively, and $\bar{y}$ is the mean of observed values.

Table 21: The R^2, MSE, RMSE, and ME for different double sigmoid growth models.

Model	Method	R^2	MSE	RMSE	ME
Double Burr Type XII-logistic	NLS	0.96455	1.33×10^{-7}	0.00037	0.96463
	ML	0.96082	1.47×10^{-7}	0.00056	0.96094
Modified double Burr Type XII-logistic	NLS	0.96486	1.32×10^{-7}	0.00036	0.96492
	ML	0.96372	1.36×10^{-7}	0.00037	0.96372
Double logistic	NLS	0.96357	1.40×10^{-7}	0.00037	0.96231
	ML	0.96224	1.46×10^{-7}	0.00038	0.96127
Modified double logistic	NLS	0.96364	1.36×10^{-7}	0.00036	0.96347
	ML	0.96334	1.41×10^{-7}	0.00038	0.96197

From Table 21, it is found that the modified double Burr Type XII-logistic sigmoid growth model is the appropriate model with the largest R^2, ME, and the smallest MSE, RMSE.

In order to compare the proposed models with the existing models, the *corrected Akaike information criterion* (AICc) is used. Also, the *likelihood ratio test* (LRT) is used to study the significance of the parameters for these models. The results of AICc and LRT are given in Table 22.

Table 22: The AICc and *p-values* of the LRT test for different double sigmoid growth models.

Model	AICc	*p-value*
Double Burr Type XII-logistic	-2302.885	2.2×10^{-16}
Modified double Burr Type XII-logistic	$\mathbf{-2304.322}$	$\mathbf{2.2\times 10^{-16}}$
Double logistic	-2291.046	2.2×10^{-16}
Modified double logistic	-2301.468	2.2×10^{-16}

As observed from Table 22, it is clear that the LRT is significant ($p - value < 0.05$) for all different double sigmoid growth models and the modified double Burr Type XII-logistic sigmoid growth model is the best model for describing these data with the smallest value of AICc.

6. Conclusions

In this paper, new double sigmoidal growth curves were proposed based on the Burr Type XII distribution. In addition, for modeling the new proposed curves, the procedure of summation of two single sigmoidal growth curves was considered. Estimating the parameters of the new proposed models was provided by NLE and ML estimation methods. The performance of the proposed models was evaluated through a simulation study. The evaluation was based on the RAB, REMSE, and the $\mathrm{A.C.I}$ for each estimate. The COVID-19 data set was analyzed to demonstrate how the proposed models can be used in practice. The results showed that the new proposed model, the modified double Burr Type XII–logistic sigmoid growth model is superior over the other models with respect to R^2, MSE, RMSE, ME, and AICc.

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