Bayesian Estimation and Prediction for Discrete Alpha Power Inverted Kumaraswamy Distribution

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Abstract

This paper discusses the Bayesian estimation for the parameters, survival, hazard rate and alternative hazard rate functions of the three unknown parameter of the discrete alpha power inverted Kumaraswamy distribution when the lifetimes are Type-II censored. The independent exponential prior for the alpha power parameter and the joint bivariate prior for the shape parameters of the inverted Kumaraswamy distribution is used to obtain the posterior distributions. The estimators are derived under squared error and linear-exponential loss functions. Credible intervals for the parameters, survival, hazard rate and alternative hazard rate functions are obtained. Bayesian prediction (point and interval) for the future observation is investigated under the two-sample prediction Scheme. The efficiency of the Bayes estimates is investigated, through some measurements of accuracy for different sample sizes. Regarding the results of the simulation study, it seems to perform better when the sample size increases and the level of censoring decreases. Also, in most cases the results under the linear-exponential loss function is better than the corresponding results under squared error loss function. Two real data sets are applied to ensure the theoretical results and confirm its

applicability to real life applications. to compare the efficiency of these estimators under different loss functions.

Keywords: *Alpha power inverted Kumaraswamy distribution; Bayes estimators; Squared error loss function; Linear-exponential loss function; Credible intervals; Bayesian prediction; Monte Carlo simulation.*

1. Introduction

Recently, probability distributions play an important role in modeling naturally occurring phenomena. In fact, the statistics literatures accommodate many of continuous distributions and their successful applications. However, there still remain many real-world phenomena involving data, which do not follow any of the traditional probability distributions. So, several attempts are introduced by many researchers to provide more flexibility to a family of distributions; for example, Marshall and Olkin (1997), Eugene *et al.* (2002), Cordeiro and Castro (2011) and Alzaatreh *et al.* (2013). For more details about methods of generating distributions see, Lee *et al.* (2013) and Jones (2015).

Mahdavi and Kundu (2016) added an extra parameter to a family of distributions functions to let the given family more flexible. They called the new method *α-power transformation* (APT) method, which can be used quite effectively for purposes of data analysis. They proposed α-power exponential distribution which has desirable properties, such as the *cumulative distribution function* (cdf) is appropriate for analyzing censored data since it can be written in explicit form. Also, the *probability density function* (pdf) and *hazard rate function* (hrf) of α-power exponential distribution acts like Weibull, Gamma or generalized exponential distributions.

The cdf, $F(x)$ of the APT, which is the cdf of a continuous random variable X, for $X \in \mathbb{R}$, is defined as follows:

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}, & \alpha > 0, \alpha \neq 1, \\ F(x)\,, & \alpha = 1, \end{cases} \tag{1}$$

and the corresponding pdf is

$$f_{APT}(x) = \begin{cases} \frac{\ln(\alpha)}{\alpha-1} f(x)\alpha^{F(x)}, & \alpha > 0, \alpha \neq 1, \\ f(x)\,, & \alpha = 1, \end{cases} \tag{2}$$

where α is a shape parameter.

Many researchers applied the APT method to many distributions, such as Nassar *et al.* (2017) presented the alpha power Weibull distribution. Dey *et al.* (2017) introduced the alpha power generalized exponential distribution. Nadarajah and Okorie (2018) studied the moment properties of the alpha power generalized exponential distribution. Mead *et al.* (2019) obtained some statistical properties of the APT family and considered the alpha power exponentiated Weibull distribution. Also, Nassar *et al.* (2020) discussed the parameter estimation of the alpha power exponential distribution using nine methods of estimation.

Although it is common in reliability lifetime modeling, to deal with failure data as continuous, indicating some degree of accuracy in measurement, practically; failures are observed at fixed inspection intervals, happened in a discrete procedure or are simply recorded in boxes. In survival analysis, the *survival function* (sf) may be a function of discrete random variable which is a discrete version of base continuous random variable. Such as, the survival time of patients is counted by number of days or weeks or the length of stay in an observation ward is counted by number of days. Also, in real life, the reliability data are measured in terms of the numbers of runs, cycles or shocks the device sustains before it fails. For example, the number of times the devices are switched on/off, the lifetime of the switch is a discrete random variable. Also, the number of voltage fluctuations; which an electrical or electronic item can withstand before its failure, the life of equipment is measured by the number of completed cycles or the number of times it operated before failure, or the life of weapon is measured by the number of rounds fired prior to failure.

Some known discrete distributions such as geometric, Poisson, binomial, beta binomial, multinomial, hypergeometric, negative binomial etc., have limited applicability as models for failure times, reliability and counts. Therefore, it is realistic and suitable to model the discrete failure time by an appropriate discrete lifetime distribution generated from the base continuous distribution keeping one or more important characters of the continuous distribution. [For more details see, Lai (2013) and Chakraborty and Chakravorty (2016)].

Several discrete lifetime distributions are constructed by discretizing their conjugate continuous models using several methods. [see, Bracquemond and Gaudoin (2003) and Chakraborty (2015)]. The general approach of discretization of some known continuous distributions have been attracting great interest for use as lifetime distributions by many researchers [see, Nakagawa and Osaki (1975), Khan *et al.* (1989), Inusah and Kozubowski (2006), Krishna and Pundir (2009), Jazi *et al.* (2010), Gomez-Deniz and Calderin-Ojeda (2011) and Nekoukhou *et al*. (2012)]. Recently, many authors considered Bayesian estimation and prediction for several discrete distributions. [see Migdadi (2014), Kamari *et al.* (2016), Ashour and Muiftah (2019), Hegazy *et al*. (2021), and El-Morshedy *et al.* (2021)].

AL-Dayian *et al*. (2023) derived *Discrete alpha power inverted Kumaraswamy* (DAIKum) distribution and is denoted by DAPIKum (α, λ, β) distribution using the general approach of discretization of a continuous distribution. They obtained some important distributional, reliability properties and ML estimators for the DAPIKum distribution. The *probability mass function* (pmf) of DAIKum distribution is given by

$$P(x; \alpha, \lambda, \beta) \equiv P(x) = \frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - 1}, \quad x = 0, 1,2, \dots, \quad \alpha, \lambda, \beta > 0, \alpha \neq 1. \tag{3}$$

The corresponding cdf, sf, hrf and *alternative* hrf (ahrf), respectively, are as follows:

$$F(x; \alpha, \lambda, \beta) \equiv F(x) = P(X \leq x) = \frac{\alpha^{(1-\left(2+x\right)^{-\lambda})^{\beta}} - 1}{\alpha - 1}, \ \alpha \neq 1, \tag{4}$$

$$S(x; \alpha, \lambda, \beta) \equiv S(x) = P(X \geq x) = \frac{\alpha - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - 1}, \quad x = 0, 1, 2, \dots, \quad \alpha \neq 1, \tag{5}$$

$$h(x; \alpha, \lambda, \beta) \equiv h(x) = \frac{P(x)}{S(x)} = \frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}, \tag{6}$$

and

$$ah(x; \alpha, \lambda, \beta) \equiv ah(x) = ln\left[\frac{S(x)}{S(x+1)}\right]$$

$$= ln\left[\frac{\alpha - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - \alpha^{(1-(2+x)^{-\lambda})^{\beta}}}\right], \ x = 0, 1,2, \dots, \quad \alpha, \lambda, \beta > 0, \ \alpha \neq 1. \tag{7}$$

2. Bayesian Estimation

Bayesian estimation approach has received a lot of attention in recent times for analyzing failure time data, which has mostly been proposed as an alternative to that of the traditional methods. When prior knowledge about the parameter is not available, it is possible to make use of the non-informative prior in Bayesian analysis. But when the prior knowledge is available, it is better to use the informative prior than the non-informative. In such a context, the estimation of unknown parameters of the DAPIKum (α, λ, β) will be discussed under the Bayesian framework.

Suppose that $X_{(1)} \le X_{(2)} \le \cdots \le X_{(r)}$ is a Type-II censored sample of size r obtained from a life-test on n items whose lifetimes have DAPIKum distribution, and then the likelihood function is

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} P\left(x_{(i)}\right)\right\}\left[S\left(x_{(r)}\right)\right]^{n-r}, \tag{8}$$

where $P(x)$ and $S(x)$are given, respectively, by (3) and (5).

The $x_{(i)}{}'$s are ordered times for $i = 1,2, \ldots r$.

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} \frac{\alpha^{\left(1-\left(2+x_i\right)^{-\lambda}\right)^{\beta}} - \alpha^{\left(1-\left(1+x_i\right)^{-\lambda}\right)^{\beta}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{\left(1-\left(1+x_r\right)^{-\lambda}\right)^{\beta}}}{\alpha-1}\right\}^{n-r}, \tag{9}$$

which can be written as

$$L\left(\underline{\varphi}|\underline{x}\right) \propto \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}} - \alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}, \tag{10}$$

where

$$w_{i1} = (1-(1+x_i)^{-\lambda})^{\beta}, \qquad w_{i2} = (1-(2+x_i)^{-\lambda})^{\beta}$$

and $\quad w_r = (1-(1+x_r)^{-\lambda})^{\beta}.$ (11)

In a Bayesian framework, the unknown parameters in any model are handled as random variables rather than fixed constants. This is a reasonable assumption because the parameters of any population cannot remain constant throughout the research. Assuming prior distributions of unknown parameters as independent prior density for α as $\alpha \sim Exponentil\,(d)$ and λ, β has joint bivariate prior density function that was used by AL-Hussaini and Jaheen (1992).

The priors of α, λ, β are

$$\pi_1(\alpha) \propto e^{-d\alpha},\; d, \alpha > 0\,, \alpha \neq 1 \tag{12}$$

$$\pi_2(\lambda, \beta) = g_1(\lambda|\beta)g_2(\beta)\;\; , \lambda, \beta > 0, \tag{13}$$

where

$$g_1(\lambda|\beta) = \frac{\beta^a}{\Gamma(a)}\lambda^{a-1}e^{-\beta\lambda},\;\; \lambda, \beta, a > 0, \alpha \neq 1 \tag{14}$$

and

$$g_2(\beta) = \frac{b^c}{\Gamma(c)}\,\beta^{c-1}\,e^{-b\beta},\; \beta, b, c > 0. \tag{15}$$

The joint prior density of α, λ and β can be written as

$$\pi\left(\underline{\varphi}\right) \propto \lambda^{a-1}\beta^{a+c-1}\,e^{-\beta(\lambda+b)-d\alpha},$$

$$\alpha, \lambda, \beta, a, b, c, d > 0, \alpha \neq 1 \tag{16}$$

where $\underline{\varphi}$ is the vector of parameters α, λ and β and a, b, c and d are the hyper parameters which assumed to be chosen in which have minimal or no effect on posterior distribution

The joint posterior distribution $\underline{\varphi}$ can be derived by combining the likelihood function in (10) and the joint prior distribution in (16) as follows:

$$\pi\left(\underline{\varphi}|\underline{x}\right) \propto L\left(\underline{\varphi};\underline{x}\right)\pi\left(\underline{\varphi}\right), \tag{17}$$

$$\pi\left(\underline{\varphi}|\underline{x}\right) = k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\, \left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}, \tag{18}$$

where $k^{-1} = \int_{\underline{\varphi}} L\left(\underline{\varphi};\underline{x}\right)\pi\left(\underline{\varphi}\right)\, d\underline{\varphi}$,

$$= \int_{\underline{\varphi}} \lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\, \left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}\, d\underline{\varphi}, \tag{19}$$

which is a normalizing constant, $\int_{\underline{\varphi}} = \int_{\alpha}\int_{\lambda}\int_{\beta}$ and $d\underline{\varphi} = d\alpha\, d\lambda\, d\beta$.

The marginal posterior distributions of the parameters $\underline{\varphi}$ can be expressed as follows:

$$\pi\left(\varphi_i|\underline{x}\right) = \int_{\underline{\varphi}_j} \pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}_j,\;\; i \neq j\;,\;\; i,j = 1,2,3. \tag{20}$$

a. Point estimation

The Bayes point estimators of the parameters, sf, hrf and ahrf are considered based on informative prior and two different loss functions: *squared error* (SE) and *linear-exponential* (LINEX) loss functions.

I. Bayesian estimation under squared error loss function

One of the most used loss functions is the SE loss function. It is commonly used with the Bayesian estimation because it does not need extensive numerical computation. It is a symmetric loss function that assigns equal weight to overestimation as well as underestimation. The Bayes estimator under SE loss function is the posterior mean. Thus, the Bayes estimator of $g(\alpha,\lambda,\beta)$ (any function of parameters α,λ,β) under the SE loss function, can be derived as

$$\hat{g}_{(SE)}(\alpha,\lambda,\beta) = \int_{\underline{\varphi}} g(\alpha,\lambda,\beta)\, \pi\big(\alpha,\lambda,\beta|\underline{x}\big)\, d\underline{\varphi}. \tag{21}$$

Under the SE loss function, the Bayes estimators of each α,λ and β are the means of their marginal posteriors and are defined by

$$\hat{\varphi}_{i(SE)} = E\big(\varphi_i|\underline{x}\big) = \int_{\underline{\varphi}} \varphi_i\, \pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}$$

$$= \int_{\underline{\varphi}} \varphi_i\, k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha} \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi},$$

$$\varphi_1,\varphi_2,\varphi_3 > 0, \varphi_i \neq 1\,, i = 1,2,3, \tag{22}$$

where $\varphi_1 = \alpha,\ \varphi_2 = \lambda\ and\ \varphi_3 = \beta.$

Since the posterior distributions in (22) cannot be obtained in explicit form, therefore, numerical methods should be used.

Also, the Bayes estimators of the sf, hrf and ahrf under SE loss function can be obtained using (5)-(7) and (18) as follows:

$$\hat{S}_{(SE)}(x_0) = E\big(S(x_0)|\underline{x}\big)$$

$$= k\int_{\underline{\varphi}} \frac{\alpha - \alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha - 1}\, \lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}$$

$$\times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}, \tag{23}$$

$$\hat{h}_{(SE)} = E\big(h(x_0)|\underline{x}\big)$$

$$= k\int_{\underline{\varphi}} \frac{\alpha^{(1-(2+x_0)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha - \alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}\, \lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}$$

$$\times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}, \tag{24}$$

and

$$\widehat{ah}_{(SE)} = E\left(ah(x_0)\middle|\underline{x}\right)$$

$$= k\int_{\underline{\varphi}} ln\left[\frac{\alpha-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha-\alpha^{(1-(2+x_0)^{-\lambda})^{\beta}}}\right]\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}\, d\underline{\varphi}. \qquad (25)$$

II. Bayesian estimation under linear-exponential loss function

The SE loss function as a symmetric loss function has probably been the most popular loss function, that gives equal weight to over- and under- estimation of the parameters under consideration. Therefore, the use of a symmetric loss function might be inappropriate for different estimation problems. However, in life testing, over estimation may be more serious than under estimation or vice versa.

Under the LINEX loss function, the Bayes estimators for the parameters α, λ and β are given, respectively, by

$$\widehat{\varphi}_{i(LINEX)} = \left(\frac{-1}{v}\right) ln\left\{E(e^{-v\varphi_i}|\underline{x})\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left\{\int_{\underline{\varphi}} e^{-v\varphi_i}\,\pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left\{\int_{\underline{\varphi}} e^{-v\varphi_i}\, k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\right.$$

$$\left.\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}\right\},$$

$$\varphi_1, \varphi_2, \varphi_3 > 0, \varphi_i \neq 1\,, i = 1,2,3, \qquad (26)$$

where v is constant and $v \neq 0$.

Similarly, the Bayes estimators of the sf, hrf and ahrf under LINEX loss function can be obtained using (5)-(7) and (18) as follows:

$$\hat{S}_{(LINEX)}(x_0) = \left(\frac{-1}{v}\right) ln\left\{E(e^{-vs(x_0)}|\underline{x})\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left[\int_{\underline{\varphi}} e^{-v\left(\frac{\alpha-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha-1}\right)} k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\right.$$

$$\left.\times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}\right], \tag{27}$$

$$\hat{h}_{(LINEX)}(x_0) = \left(\frac{-1}{v}\right) ln\left\{E\left(e^{-vh(x_0)}|\underline{x}\right)\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left[\int_{\underline{\varphi}} e^{-v\left(\frac{\alpha^{(1-(2+x_0)^{-\lambda})^{\beta}}-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}\right)} k\lambda^{a-1}\beta^{a+c-1}\right.$$

$$\left.\times e^{-\beta(\lambda+b)-d\alpha} \times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}\right], \tag{28}$$

$$\widehat{ah}_{(LINEX)}(x_0) = \left(\frac{-1}{v}\right) ln\left\{E\left(e^{-vah(x_0)}|\underline{x}\right)\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left[\int_{\underline{\varphi}} e^{-v\left(ln\left[\frac{\alpha-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}{\alpha-\alpha^{(1-(2+x_0)^{-\lambda})^{\beta}}}\right]\right)} k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\right.$$

$$\left.\times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi}\right]. \tag{29}$$

The Bayes estimates of the parameters, sf, hrf and ahrf can be obtained by evaluating (22)- (29) numerically.

b. Credible interval

In general, a two-sided $100(1-\omega)$ % credible interval of the parameters $\underline{\varphi}$ is given by

$$P\left[L(\underline{x}) < \varphi_i < U(\underline{x})\right] = \int_{L(\underline{x})}^{U(\underline{x})} \pi(\varphi_i|\underline{x})\, d\underline{\varphi} = 1-\omega, \tag{30}$$

where $L(x)$ and $U(x)$, are the *lower limit* (LL) and *upper limit* (UL).

From the marginal posterior distributions of the parameters $\underline{\varphi}$ in (20), the $100(1-\omega)$ % credible interval for φ_i are given by

$$P\left[\varphi_i > L\left(\underline{x}\right)|\underline{x}\right] = 1 - \frac{\omega}{2} \quad \text{and} \quad P\left[\varphi_i > U\left(\underline{x}\right)|\underline{x}\right] = \frac{\omega}{2}. \qquad (31)$$

Then a $100\ (1-\omega)$ % credible interval for α is $(L\ (\underline{x}), U\ (\underline{x}))$, is given by

$$P\left[\alpha > L\left(\underline{x}\right)|\underline{x}\right] = k\int_{L(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\lambda d\beta d\alpha = 1-\frac{\omega}{2}, \quad (32)$$

$$P\left[\alpha > U\left(\underline{x}\right)|\underline{x}\right] = k\int_{U(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\lambda d\beta d\alpha = \frac{\omega}{2}, \quad (33)$$

Also, a 100 (1- ω) % credible interval for λ, β is $(L\ (\underline{x}), U\ (\underline{x}))$ and can be obtained respectively as follows:

$$P\left[\lambda > L\left(\underline{x}\right)|\underline{x}\right] = k\int_{L(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\alpha d\beta d\lambda = 1-\frac{\omega}{2}, \quad (34)$$

$$P\left[\lambda > U\left(\underline{x}\right)|\underline{x}\right] = k\int_{U(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\alpha d\beta d\lambda = \frac{\omega}{2}, \quad (35)$$

and

$P\left[\beta > L(\underline{x})|\underline{x}\right]$

$$= k\int_{L(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\alpha d\lambda d\beta = 1-\frac{\omega}{2}, \quad (36)$$

$P\left[\beta > U(\underline{x})|\underline{x}\right]$

$$= k\int_{U(\underline{x})}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\alpha d\lambda d\beta = \frac{\omega}{2}, \quad (37)$$

Furthermore, a 100 (1- ω) % credible interval for S(x), h(x) and ah(x) are given respectively as follows:

$$P\left[L(\underline{x}) < S < U(\underline{x})\right] = \int_{L(\underline{x})}^{U(\underline{x})}\pi(S|\underline{x})\, dS = 1-\omega, \quad (38)$$

$$P\left[L(\underline{x}) < h < U(\underline{x})\right] = \int_{L(\underline{x})}^{U(\underline{x})}\pi(h|\underline{x})\, dS = 1-\omega, \quad (39)$$

and

$$P\left[L(\underline{x}) < ah < U(\underline{x})\right] = \int_{L(\underline{x})}^{U(\underline{x})}\pi(ah|\underline{x})\, dS = 1-\omega, \quad (40)$$

where $\pi(S|\underline{x}), \pi(h|\underline{x})$ and $\pi(ah|\underline{x})$ re the posterior distribution of sf, hrf and $a$hrf which can be obtained numerically.

To obtain the two-sided a 100 $(1-\tau)$ % credible interval for $\underline{\varphi}$, sf, hrf and $a$hrf (32) to (40) should be solved numerically.

3. Bayesian Prediction Based on Two-Sample Prediction

Considering that $X = (X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)})$ are the first r ordered life times in a random sample of n components(Type II censoring) whose failure times are identically distributed as DAPIKum (α, λ, β) distribution; informative sample, and $Y = (Y_1 \leq Y_2 \leq \cdots \leq Y_m)$ is a second independent random sample of size m of future observables from the same distribution; the future sample. Our aim is to predict the s^{th}order statistic in the future sample based on the informative sample. For the future sample of size m, let $Y_{(s)}$ denotes the s^{th} order statistic, $1 \leq s \leq m$. The conditional pmf of $Y_{(s)}$can be obtained as follows:

$$P\left(Y_{s:m} = y_{(s)} | \underline{\varphi}\right) = \frac{m!}{(s-1)!(m-s)!} \int_{F\left(y_{(s)}-1\right)}^{F\left(y_{(s)}\right)} v^{s-1}(1-v)^{m-s} dv$$

$$= \frac{m!}{(s-1)!(m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j}$$

$$\times \left[\left[F\left(y_{(s)}\right)\right]^{s+j} - - \left[F\left(y_{(s)} - 1\right)\right]^{s+j}\right]$$

$$= \frac{m!}{(s-1)!(m-s)!} \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j}$$

$$\times \left[\left[\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^{s+j} - \left[\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^{s+j}\right],$$

$$y_{(s)} = 0,1,2, \ldots \qquad (41)$$

Assuming that $\underline{\varphi} = (\alpha, \lambda, \beta)$ are unknown, then the *Bayesian* predictive *mass function* (BPMF) of $Y_{(s)}$ given $\underline{x}$ is given by

$$P^*\left(y_{(s)} \middle| \underline{x}\right) = \int_{\underline{\varphi}} P\left(y_{(s)} \middle| \underline{\varphi}\right) \pi\left(\underline{\varphi} \middle| \underline{x}\right) d\underline{\varphi},$$

$$y_{(s)} = 0,1, \ldots, s = 1, 2, 3, \ldots, m. \qquad (42)$$

Therefore, The BPMF of $Y_{(s)}$ given $\underline{x}$ can be obtained by substituting (18) and (41) into (42), then the BPMF of the future order statistic$Y_{(s)}, s = 1, 2, 3, \ldots, m,$ is given by

$$P^*\left(y_{(s)}\middle|\underline{x}\right) = \frac{m!}{(s-1)!(m-s)!}\int_0^\infty\int_0^\infty\int_0^\infty \left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\right.$$

$$\times\left[\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^{s+j} - \left.\left[\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^{s+j}\right\}\times k\lambda^{a-1}\beta^{a+c-1}\,e^{-\beta(\lambda+b)-d\alpha}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}\; d\underline{\varphi},$$

where $\underline{\varphi} > 0,\; y_{(s)} = 0,1,\dots, s = 1,2,3,\dots,m.$ (43)

a. Point prediction

Based on Type II censoring, two-sample Bayesian prediction is considered under two types of loss functions SE loss function, as a symmetric loss function, and LINEX loss function, as an asymmetric loss function.

I. Bayesian prediction based on squared error loss function

The *Bayes predictor* (BP) for the future observation $Y_{(s)}$, under the SE loss function can be derived using (43) as follows:

$$\hat{y}_{(s)(SE)} = E\left(Y_{(s)}\middle|\underline{x}\right) = \sum_{y_{(s)}=0}^{\infty} y_{(s)}\, P^*\left(y_{(s)}\middle|\underline{x}\right)$$

$$= \frac{m!}{(s-1)!(m-s)!}\sum_{y_{(s)}=0}^{\infty} y_{(s)}\int_0^\infty\int_0^\infty\int_0^\infty\left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\right.$$

$$\times\left[\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^{s+j} - \left.\left[\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^{s+j}\right\} k\lambda^{a-1}\beta^{a+c-1}\,e^{-\beta(\lambda+b)-d\alpha}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}\; d\underline{\varphi},$$

$y_{(s)} = 0,1,\dots, s = 1,2,3,\dots,m.$ (44)

II. **Bayesian prediction based on linear-exponential loss function**

The BP for the future observation $Y_{(s)}$, under the LINEX loss function can be obtained using (43) as given below

$$\hat{y}_{(s)(\text{LINEX})} = \left(\frac{-1}{v}\right) ln\left[E\left(e^{-vy_{(s)}}\middle|\underline{x}\right)\right] = \sum_{y_{(s)}=0}^{\infty} e^{-vy_{(s)}} P^*\left(y_{(s)}\middle|\underline{x}\right)$$

$$= \left(\frac{-1}{v}\right)\frac{m!}{(s-1)!(m-s)!} ln\left[\sum_{y_{(s)}=0}^{\infty} e^{-vy_{(s)}} \int_0^\infty \int_0^\infty \int_0^\infty \left\{\sum_{j=0}^{m-s}\binom{m-s}{j}\right.\right.$$

$$\left.\times (-1)^j \frac{1}{s+j}\left[\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^{s+j} - \left[\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^{s+j}\right\}$$

$$\left.\times k\lambda^{a-1}\beta^{a+c-1} e^{-\beta(\lambda+b)-d\alpha} \left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r}\right] d\underline{\varphi},$$

$$y_{(s)} = 0,1,\dots, s = 1,2,3,\dots,m. \tag{45}$$

Special cases:

I. If $s = 1$, in (44) and (45), one can predict the minimum observable, $Y_{(1)}$; which represents the first failure time in the future sample of size m, given that *r* components had already failed in the informative sample of size *n*. Hence, under the SE and LINEX loss functions, one gets

$$\hat{y}_{(1)(SE)} = \sum_{y_{(1)}=0}^{\infty} y_{(1)} \int_0^\infty \int_0^\infty \int_0^\infty k\lambda^{a-1}\beta^{a+c-1} e^{-\beta(\lambda+b)-d\alpha}$$

$$\times \left[1-\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^m - \left[1-\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^m \left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi},$$

$$y_{(1)} = 0,1,\dots, \tag{46}$$

and

$\hat{y}_{(1)(\text{LINEX})}$

$$= \left(\frac{-1}{v}\right) ln \left[\sum_{y_{(1)}=0}^{\infty} e^{-vy_{(1)}} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} k\lambda^{a-1} \beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \right.$$

$$\left. \left[1 - \frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^m - \left[1 - \frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^m \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}} - \alpha^{w_{i1}}}{\alpha-1}\right\} \left\{\frac{\alpha - \alpha^{w_r}}{\alpha-1}\right\}^{n-r} \right] d\underline{\varphi},$$

$$y_{(1)} = 0,1, \dots. \tag{47}$$

II. If $s = 1$, in (44) and (45), one can predict the maximum observable, $Y_{(m)}$; which represents the largest failure time in the future sample of size m, given that r components had already failed in the informative sample of size n. Hence, under the SE and LINEX loss functions, one obtains

$$\hat{y}_{(m)(SE)} = \sum_{y_{(m)}=0}^{\infty} y_{(m)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \times \left[\frac{\alpha^{w_{i2}} - 1}{\alpha - 1}\right]^m - \left[\frac{\alpha^{w_{i1}} - 1}{\alpha - 1}\right]^m$$

$$\times\, k\lambda^{a-1} \beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha}\ \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}} - \alpha^{w_{i1}}}{\alpha-1}\right\} \left\{\frac{\alpha - \alpha^{w_r}}{\alpha-1}\right\}^{n-r}\ d\underline{\varphi},$$

$$y_{(m)} = 0,1, \dots, \tag{48}$$

and

$$\hat{y}_{(m)(\text{LINEX})} = \left(\frac{-1}{v}\right) ln \left[\sum_{y_{(m)}=0}^{\infty} y_{(m)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-vy_{(m)}} \right.$$

$$\times\, k\lambda^{a-1} \beta^{a+c-1}\ e^{-\beta(\lambda+b)-d\alpha} \left[\frac{\alpha^{w_{i2}-1}}{\alpha-1}\right]^m - \left[\frac{\alpha^{w_{i1}-1}}{\alpha-1}\right]^m$$

$$\left. \times \left\{\prod_{i=1}^{r} \frac{\alpha^{w_{i2}} - \alpha^{w_{i1}}}{\alpha-1}\right\} \left\{\frac{\alpha - \alpha^{w_r}}{\alpha-1}\right\}^{n-r} \right] d\underline{\varphi},$$

$$y_{(m)} = 0,1, \dots. \tag{49}$$

b. Bayesian predictive bounds

A 100 (1-ω)% *Bayesian predictive bounds* (BPB) for the future observation $Y_{(s)}$, such that

$$P\left[L_{(s)}(\underline{x}) < y_{(s)} < U_{(s)}(\underline{x})|\underline{x}\right] = \sum_{L_{(s)}(\underline{x})}^{U_{(s)}(\underline{x})} P^*\left(y_{(s)}|\underline{x}\right) = 1 - \omega. \tag{50}$$

The lower and upper bounds $\left[L_{(s)}(\underline{x}), U_{(s)}(\underline{x})\right]$ can be obtained by evaluating

$$P\left[y_{(s)} > L_{(s)}(\underline{x})|\underline{x}\right] = \sum_{L_{(s)}(\underline{x})}^{\infty} P^*\left(y_{(s)}|\underline{x}\right)$$

$$= \frac{m!}{(s-1)!\,(m-s)!}\int_0^\infty\int_0^\infty\int_0^\infty \left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j \frac{1}{s+j}\right.$$

$$\left.\times k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\left[\frac{\alpha^{w_{i2}}-1}{\alpha-1}\right]^{s+j} - \left[\frac{\alpha^{w_{i1}}-1}{\alpha-1}\right]^{s+j}\right\}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi} = 1-\frac{\omega}{2},$$

$$y_{(s)} = 0,1,\dots, s = 1,2,3,\dots,m, \tag{51}$$

$$P\left[y_{(s)} > U_{(s)}(\underline{x})|\underline{x}\right] = \sum_{U_{(s)}(\underline{x})}^{\infty} P^*\left(y_{(s)}|\underline{x}\right)$$

$$= \frac{m!}{(s-1)!\,(m-s)!}\int_0^\infty\int_0^\infty\int_0^\infty \left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j \frac{1}{s+j}\right.$$

$$\left.\times k\lambda^{a-1}\beta^{a+c-1}\, e^{-\beta(\lambda+b)-d\alpha}\left[\frac{\alpha^{w_{i2}}-1}{\alpha-1}\right]^{s+j} - \left[\frac{\alpha^{w_{i1}}-1}{\alpha-1}\right]^{s+j}\right\}$$

$$\times\left\{\prod_{i=1}^{r}\frac{\alpha^{w_{i2}}-\alpha^{w_{i1}}}{\alpha-1}\right\}\left\{\frac{\alpha-\alpha^{w_r}}{\alpha-1}\right\}^{n-r} d\underline{\varphi} = \frac{\omega}{2},$$

$$y_{(s)} = 0,1,\dots, s = 1,2,3,\dots,m. \tag{52}$$

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of the Bayesian estimation and prediction based on simulated and real data.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates based on generated data from the DAIKum distribution. Bayes averages and credible intervals of the parameters, sf, hrf and ahrf based on Type II censoring are computed. Also, two sample Bayes predictors (point and interval) for a future observation from the DAIKum distribution based on Type II censored data are computed. Simulation studies are performed using R programming language for illustrating the estimation and prediction results obtained.

Simulation algorithm

- A combination of the population parameter values for α, λ and β are used to generate several data sets from DAIKum distribution. Also, for samples of size *n*=30, 60 and 120 using *number of replications* (NR)=10000 for each sample size.
- The samples are drawn from the population distribution, in the complete sample case and when the data are censored at the 60% and 80% level, for each sample size.
- The random samples are generated from the DAPIKum (α, β, λ) distribution using the following transformation

$$x_i = \left[\left[1 - \left[\frac{ln[(\alpha-1)u_i+1]}{ln(\alpha)}\right]^{1/\beta}\right]^{-1/\lambda} - 2\right], i = 1, 2, \dots, n,$$

where u_i are random samples from the uniform distribution (0,1), and then taking the ceiling.

- After the estimates $\hat{\alpha}$, $\hat{\lambda}$ *and* $\hat{\beta}$ are obtained, then the sf, hrf and ahrf for different values are estimated.
- Some measurements of accuracy are considered to evaluate the performance of the Bayes estimators, $\hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{S}(x_0), \hat{h}(x_0)$ and $\widehat{ah}(x_0)$. In order to study the variation and precision of the Bayes estimates is studied through the variance, the *relative absolute bias*
- *Relative absolute biases*(RAB)

  $$= \frac{|estimate - population\ parameter|}{population\ parameter},$$

  and

  the *relative error* $(RE) = \frac{\sqrt{mean\ square\ error(estimate)}}{population\ paramater}$.

The results are displayed in Tables 1-3.

4.2 Applications

In this subsection, two real data sets are provided to demonstrate the importance of the DAIKum distribution and how it can be used in real life. The estimates and their corresponding *standard errors* (se) of the parameters, sf, hrf, ahrf for the two real data sets are given in Tables 5 and 6.

Application 1

The first data set is considered by Lawless (2011). This dataset is the failure times for a sample of 15 electronic components in an acceleration life test. The data observations are: 1.0, 5.0, 6.0, 11.0, 12.0, 19.0, 20.0, 22.0, 23.0, 31.0, 37.0, 46.0, 54.0, 60.0 and 66.0.

Application 2

The second data set of this application is obtained from

Freireich *et al.* (1963). It represents the remission times (in weeks) for 21 patients who treated with placebo from 97 patients with acute leukemia participated in a clinical trial investigating the effect of 6-mercaptopurine. The remission times for the n = 21 patients treated with placebo were 1, 1, 2, 2, 3, 4, 4, 5, 5, 8, 8, 8, 8, 11, 11, 12, 12, 15, 17, 22, and 23 weeks.

Kolmogorov-Smirnov goodness of fit test is performed for each data set, to check the validity of the fitted model, and the p values are, respectively, 0.6781 and 0.5860. In each case, the p value shows that the model fits the data very well.

4.3 Concluding remarks

- It is noticed from Tables 1,2 and 3 that the RABs and REs of the Bayes averages; for the parameters, sf, hrf and ahrf, under LINEX loss function have less values than the corresponding results under the SE loss function.
- As expected, it is observed that better estimates are obtained when the sample sizes increases and level of censoring decreases, which is obvious from comparing the RABs and REs of the estimates. This confirms that more information provided by the sample, increases the accuracy of the estimates.
- The two-sided 95% credible intervals distribution become narrower as the sample size increases.
- Regarding the results in Tables 4 and 7, both Bayes predictive estimates and credible intervals of the future

observations; for simulated and real data, under SE and LINEX loss functions are very close.

- Also, the lengths of the credible intervals under LINEX loss function are shorter than the lengths under SE loss function.

References

AL-Hussaini, E. K. and Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure rate functions of the Burr Type XII failure model. *Journal of Statistical Computation and Simulation*, 41: 31-40.

Almetwally, E. M., Almongy, H. M., Saleh, H. A. (2020). Managing risk of spreading "COVID-19" in Egypt: modelling using a discrete Marshall–Olkin generalized exponential distribution. *International Journal of Probability and Statistics*, 9(2), 33-41.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A New method for generating families of continuous distributions. *Metron*, 71(1), 63-79.

Ashour, S. K. and Muiftah, M. S. A. (2019). Bayesian estimation of the parameters of discrete Weibull Type (I) distribution. *Journal of Modern Applied Statistical Methods*, 18(2): 2740.

Bracquemond, C. and Gaudoin, O. (2003). A Survey on discrete lifetime distributions. *International Journal of Reliability, Quality and Safety Engineering*, 10, 69-98.

Chakraborty, S. (2015). Generating discrete analogues of continuous probability distributions – a survey of methods and constructions. *Journal of Statistical Distributions and Applications*, 2(6), 1-30.

Chakraborty, S. and Chakravorty, D. (2016). A New discrete probability distribution with Integer support on$(-\infty, \infty)$. *Communication in Statistics –Theory and Methods*, 45(2),492-505.

Cordeiro, G.M. and de Castro, M. (2011). A New family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7), 883-898.

Dey, S., Alzaatreh, A., Zhang, C. and Kumar, D. (2017). A New extension of generalized exponential distribution with application to ozone data. *Ozone: Science & Engineering*, 39, 273-285.

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics, Theory and Methods*, 31, 497-512.

El-Morshedy, M., Eliwa, M. S. and Abhishek Tyagi. (2021). Discrete analogue of odd Weibull-G family of distributions: properties, classical and Bayesian estimation with applications to count data. *Journal of Applied Statistics*, 49(11):2928-2952. https://doi.org/10.1080/02664763.2021.1928018.

Freireich, E. J., Gehan, E., Frei-3rd, E., Schroeder, L. R., Wolman, I. J., Anbari, R., Burgert, E. O., Mills, S. D., Pinkel, D., and Selawry, O. S. (1963). The effect of 6- mercaptopurine on the duration of steroid induced remissions in acute leukemia - A model for evaluation of other potentially useful therapy, Blood. From www.bloodjournal.org by guest on April 8, 2016. For personal use only, 21, 699–716.

Mahdavi, A. and Kundu, D. (2016). A New method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13), 6543-6557.

Migdadi, H.S. (2014). Bayesian inference for the scale parameter of the discrete Rayleigh distribution. *MAGNT Research Report*, 3(2), 1073-1080.

Mead, M. E., Cordeiro, G. M., Afify, A. Z. and Al Mofleh, H. (2019). The Alpha power transformation family: properties and applications. *Pakistan Journal of Statistics and Operation Research*, 15(3), 525-545.

AL-Dayian, G. R., EL-Helbawy, A. A., Hegazy, M. A. (2023). A Discrete Analog of the Alpha Power Inverted Kumaraswamy Distribution. Submitted.

Gomez-Deniz, E. and Calderin-Ojeda, E. (2011). The Discrete Lindley distribution: properties and application. *Journal of Statistics Computation and Simulation*, 81 (11), 1405-1416.

Hegazy, M. A., EL-Helbawy, A. A., Abd EL-Kader, R. E. and AL-Dayian, G. R. (2021). Bayesian estimation and prediction of discrete Gompertz distribution. *Journal of Advances in Mathematics and Computer Science,* 36(2): 1-21.

Inusah, S. and Kozubowski, J. T. (2006). A Discrete analogue of the Laplace distribution. *Journal of Statistics Planning Inference* 136, 1090-1102.

Jazi, M. A., Lai, C. D. and Alamatsaz, M. H. (2010). A Discrete inverse Weibull distribution and estimation of its parameters. *Elsevier Statistical Methodology,* 7 (2),121- 132.

Jones, M.C. (2015). On Families of distributions with shape parameters. *International Statistical Review*, 83(2), 175-192.

Khan, M. S. A., Khalique, A. and Abouammoh, A. M. (1989). On Estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability,* 383, 348-350.

Kamari, H., Bevrani, H. and Kamary, K. (2016). Bayesian estimation of discrete Burr distribution with Two parameters. *Research and Review: Journal of Statistics and Math Ematical Sciences*, 1(2), 62-68.

Krishna, H. and Pundir, P. S. (2009). Discrete Burr and discrete Pareto distributions. *Statistical Methodology,* 6, 177-188.

Marshall, A. W. and Olkin, I. (1997). A New method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.

Lai, C.D. (2013). Issues concerning constructions of discrete lifetime models. *Quality Technology and Quantitative Management*, 10(2), 251-262.

Lee, C., Famoye, F. and Alzaatreh, A.Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *Wiley Interdisciplinary Reviews: Computational Statistics*, 5(3), 219-238.

Nadarajah, S. and Okorie, I. E. (2018). On the moments of the alpha power transformed generalized exponential distribution. *Ozone: Science & Engineering*, 40, 330-335.

Nakagawa, T. and Osaki, S. (1975). The Discrete Weibull distribution. *IEEE Transactions on Reliability*, 24 (5), 300-301.

Nassar, M., Alzaatreh, A., Mead, M. and Abo-Kasem, O. (2017). Alpha power Weibull distribution: properties and applications. *Communications in Statistics-Theory and Methods*, 46(20), 10236-10252.

Nassar, M., Afify, A. Z. and Shakhatreh, M. K. (2020). Estimation methods of alpha power exponential distribution with applications to engineering and medical data. *Pakistan Journal of Statistics and Operation Research*, 16 (1), 149-166.

Nekoukhou, V., Alamatsaz, M. H., and Bidram, H. (2012). A Discrete analog of Generalized exponential distribution. *Communication in Statistics-Theory and Methods*, 41 (11), 2000-2013.

Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. Second Edition, John Wiley & Sons, New York.

Appendix

Table 1

Bayes averages, relative absolute biases, relative errors, and 95% CIs for the parameters α, λ and β based on Type-II censoring ($NR = 10000,\ x_0 = 2$)

n	*r*	Par	SE						LINEX(ν = 0.5)					
			Average	RAB	RE	UL	LL	Length	Average	RAB	RE	UL	LL	Length
30	18	α	0.5143	0.0286	0.0822	0.5661	0.4597	0.1064	0.5090	0.0181	0.0329	0.5403	0.4814	0.0589
		λ	5.0458	0.0091	0.8416	5.0944	5.0021	0.0923	4.9551	0.0089	0.8034	4.9961	4.9211	0.0750
		β	10.0261	0.0026	0.2740	10.0610	9.9797	0.0813	10.007	0.0008	0.0237	10.0354	9.9757	0.0597
	24	α	0.4652	0.0285	0.0424	0.5073	0.4291	0.0781	0.4955	0.0089	0.0080	0.5111	0.4786	0.0324
		λ	5.0095	0.0018	0.0364	5.0357	4.9729	0.0628	4.9909	0.0018	0.0330	5.0012	4.9777	0.0234
		β	10.0192	0.0019	0.1480	10.0585	9.9892	0.0693	9.9942	0.0005	0.0131	10.0076	9.9801	0.0275
	30	α	0.4901	0.0197	0.0391	0.5061	0.4669	0.0391	0.5059	0.0057	0.0032	0.5122	0.4960	0.0162
		λ	5.0294	0.0015	0.0218	5.0495	5.0050	0.0445	5.0007	0.0005	0.0021	5.0090	4.9961	0.0129
		β	9.9715	0.0011	0.0084	10.0035	9.9446	0.0588	9.9974	0.0004	0.0077	10.0012	9.9936	0.0076
60	36	α	0.5087	0.0174	0.0302	0.5170	0.5004	0.0165	0.4985	0.0028	0.0008	0.5008	0.4960	0.0047
		λ	5.0065	0.0013	0.0171	5.0163	5.0006	0.0156	5.0053	0.0010	0.0115	5.0094	4.9998	0.0096
		β	10.0057	0.0006	0.0133	10.0153	9.9993	0.0159	9.9983	0.0002	0.0011	10.0012	9.9951	0.0061

Table 1. Continued

Bayes averages, relative absolute biases, relative errors, and 95% CIs for the parameters α, λ *and* β based on Type-II censoring
($NR = 10000,\ x_0 = 2$)

n	*r*	Par	SE						LINEX($\nu = 0.5$)					
			Average	**RAB**	**RE**	**UL**	**LL**	**Length**	**Average**	**RAB**	**RE**	**UL**	**LL**	**Length**
60	48	α	0.4929	0.0140	0.0198	0.5002	0.4861	0.0140	0.4989	0.0021	0.0005	0.5002	0.4969	0.0033
		λ	4.9958	0.0008	0.0067	5.0011	4.9894	0.0116	5.0023	0.0005	0.0022	5.0039	5.0009	0.0030
		β	9.9988	0.0001	0.0005	10.0027	9.9958	0.0068	9.9974	0.0001	0.0005	10.0000	9.9956	0.0043
	60	α	0.5012	0.0025	0.0006	0.5027	0.4995	0.0032	0.5011	0.0023	0.0005	0.5018	0.5002	0.0016
		λ	4.9975	0.0005	0.0024	5.0002	4.9947	0.0055	5.0009	0.0001	0.0003	5.0013	5.0003	0.0010
		β	10.0012	0.0001	0.0006	10.0030	9.9994	0.0036	9.9988	0.0001	0.0005	10.0004	9.9974	0.0029
120	72	α	0.5018	0.0037	0.0018	0.5034	0.4995	0.0039	0.4978	0.0024	0.0002	0.4994	0.4961	0.0033
		λ	5.0021	0.0004	0.0017	5.0057	4.9982	0.0074	5.0018	0.0003	0.0013	5.0034	4.9998	0.0036
		β	10.0033	0.0003	0.0046	10.0064	9.99910	0.0074	10.0024	0.0002	0.0010	10.0045	10.0000	0.0045
	96	α	0.4991	0.0030	0.0010	0.5007	0.4978	0.0029	0.4999	0.0001	8.0285×10^{-5}	0.5009	0.4984	0.0025
		λ	4.9986	0.0003	0.0007	5.0003	4.9970	0.0032	4.9994	0.0001	0.0001	5.0010	4.9983	0.0027
		β	10.0008	8.8438×10^{-5}	0.0003	10.0016	10.0001	0.0015	9.9993	6.2238×10^{-5}	0.0001	9.9998	9.9986	0.0012
	120	α	0.5000	0.0001	3.7181×10^{-6}	0.5002	0.4998	0.0004	0.4999	9.1137×10^{-5}	8.3060×10^{-7}	0.5000	0.4998	0.0002
		λ	4.9999	1.3612×10^{-5}	1.8530×10^{-6}	5.0001	4.9996	0.0005	5.0000	1.0956×10^{-7}	1.2005×10^{-10}	5.0000	4.9999	0.0001
		β	9.9999	6.3068×10^{-6}	1.5910×10^{-6}	10.0000	9.9997	0.0003	9.9999	4.9281×10^{-6}	9.7145×10^{-7}	10.0000	9.9998	0.0001

Table 2

Bayes averages, relative absolute biases, relative errors, and 95% CIs for the sf, hrf and ahrf based on Type-II censoring ($NR = 10000, x_0 = 2$)

n	*r*	Par	SE						LINEX($\nu = 0.5$)					
			Average	**RAB**	**RE**	**UL**	**LL**	**Length**	**Average**	**RAB**	**RE**	**UL**	**LL**	**Length**
30	18	$S(x_0)$	0.8786	0.0422	0.5074	0.9131	0.8452	0.06783	0.8507	0.0091	0.0239	0.8694	0.8248	0.0445
		$h(x_0)$	0.1115	0.4666	0.8122	0.2036	0.0323	0.1713	0.2032	0.0287	0.0144	0.2287	0.1812	0.0475
		$ah(x_0)$	0.2172	0.0743	0.1218	0.2870	0.1535	0.1335	0.2258	0.0378	0.0315	0.3014	0.1862	0.3014
	24	$S(x_0)$	0.8664	0.0278	0.2203	0.8893	0.8461	0.04322	0.8324	0.0039	0.0044	0.8485	0.8174	0.0311
		$h(x_0)$	0.2016	0.0361	0.0228	0.2263	0.1815	0.0448	0.2135	0.0206	0.0074	0.2158	0.2098	0.0060
		$ah(x_0)$	0.2192	0.0659	0.0957	0.2443	0.1963	0.0479	0.2378	0.0134	0.0039	0.2415	0.2327	0.0088
	30	$S(x_0)$	0.8227	0.0239	0.1633	0.8449	0.8108	0.0341	0.8456	0.0031	0.0027	0.8517	0.8385	0.0132
		$h(x_0)$	0.1987	0.0254	0.0113	0.2177	0.1811	0.0366	0.2083	0.0126	0.0028	0.2114	0.2056	0.0058
		$ah(x_0)$	0.2441	0.0401	0.0355	0.2687	0.2213	0.0474	0.2296	0.0118	0.0031	0.2344	0.2256	0.0088
60	36	$S(x_0)$	0.8570	0.0167	0.0794	0.8774	0.8292	0.0481	0.8524	0.0060	0.0106	0.8632	0.8441	0.0191
		$h(x_0)$	0.2049	0.0206	0.0742	0.2208	0.1859	0.0349	0.2020	0.0104	0.0103	0.2110	0.1930	0.0179
		$ah(x_0)$	0.2247	0.0426	0.0401	0.2457	0.2029	0.0427	0.2418	0.0302	0.0202	0.2466	0.2340	0.0126
	48	$S(x_0)$	0.8556	0.0149	0.0638	0.8783	0.8369	0.0414	0.8482	0.0031	0.0028	0.8540	0.8436	0.0104
		$h(x_0)$	0.2033	0.0194	0.0139	0.2125	0.1931	0.0194	0.2121	0.0140	0.0035	0.2171	0.2063	0.0108
		$ah(x_0)$	0.2403	0.0237	0.0124	0.2516	0.2305	0.0210	0.2384	0.0158	0.0056	0.2414	0.2344	0.0070
	60	$S(x_0)$	0.8411	0.0045	0.0059	0.8476	0.8332	0.0143	0.8391	0.0022	0.0013	0.8433	0.8333	0.01000
		$h(x_0)$	0.2067	0.0118	0.0024	0.2115	0.2014	0.0101	0.2083	0.0042	0.0003	0.2108	0.2059	0.0049
		$ah(x_0)$	0.2301	0.0194	0.0083	0.2347	0.2238	0.0109	0.2331	0.0066	0.0010	0.2357	0.2300 1	0.0056

Table 2. Continued

Bayes averages, relative absolute biases, relative errors, and 95% CIs for the sf, hrf and ahrf based on Type-II censoring ($NR = 10000,\ x_0 = 2$)

n	***r***	**Par**	**SE**						**LINEX(ν = 0.5)**					
			Average	**RAB**	**RE**	**UL**	**LL**	**Length**	**Average**	**RAB**	**RE**	**UL**	**LL**	**Length**
120	72	$S(x_0)$	0.8388	0.0049	0.0070	0.8444	0.8341	0.0103	0.8419	0.0012	0.0004	0.8456	0.8380	0.0076
		$h(x_0)$	0.2104	0.0057	0.0006	0.2159	0.2051	0.0107	0.2100	0.0040	0.0003	0.2147	0.2046	0.0100
		$ah(x_0)$	0.2234	0.0481	0.0510	0.2333	0.2149	0.0183	0.2354	0.0028	0.0002	0.2382	0.2318	0.0064
	96	$S(x_0)$	0.8398	0.0038	0.0040	0.8436	0.8367	0.0069	0.8421	0.0010	0.0003	0.8453	0.8387	0.0066
		$h(x_0)$	0.2082	0.0048	0.0004	0.2107	0.2060	0.0047	0.2087	0.0022	8.5863×10^{-5}	0.2105	0.2056	0.0049
		$ah(x_0)$	0.2315	0.0134	0.0040	0.2351	0.2269	0.0082	0.2344	0.0012	3.0661×10^{-5}	0.2378	0.2316	0.0062
	120	$S(x_0)$	0.8408	0.0025	0.0018	0.8438	0.8383	0.0055	0.8423	0.0008	0.0001	0.8432	0.8413	0.0018
		$h(x_0)$	0.2082	0.0044	3.4187×10^{-4}	0.2099	0.2069	0.0030	0.2090	0.0007	9.2114×10^{-6}	0.2097	0.2085	0.0011
		$ah(x_0)$	0.2328	0.0080	0.0014	0.2350	0.2310	0.0040	0.2344	0.0011	2.7518×10^{-5}	0.2365	0.2326	0.0039

Table 3

Bayes averages, relative absolute biases, relative errors, and 95% CIs for the parameters α, λ *and* β based on complete sample *(NR= 10000)*

n	Par	SE						LINEX(ν= 0.5)					
		Average	**RAB**	**RE**	**UL**	**LL**	**Length**	**Average**	**RAB**	**RE**	**UL**	**LL**	**Length**
50	$\alpha = 2$	2.0226	0.0113	0.2050	2.1008	1.9391	0.1617	2.0093	0.0046	0.0352	2.0221	1.9898	0.0323
	$\lambda = 3$	3.0192	0.0064	0.1486	3.0868	2.9613	0.1255	3.0058	0.0019	0.0136	3.0183	2.9946	0.0236
	$\beta = 5$	5.0383	0.0076	0.5869	5.0695	5.0017	0.0678	4.9933	0.0013	0.0178	5.0114	4.9789	0.0324
100	$\alpha = 2$	2.0053	0.0027	0.0115	2.0237	1.9895	0.0341	1.9978	0.0010	0.0019	2.0051	1.9931	0.0120
	$\lambda = 3$	2.9970	0.0009	0.0034	3.01300	2.9805	0.03246	3.0015	0.0005	0.0009	3.0052	2.9971	0.0080
	$\beta = 5$	5.0056	0.0011	0.0128	5.0348	4.9914	0.0433	5.00478	0.0010	0.0091	5.0086	5.0009	0.0076
200	$\alpha = 2$	1.9993	0.0003	0.0002	2.0013	1.9971	0.0042	1.9994	0.0002	0.0001	2.0001	1.9987	0.0014
	$\lambda = 3$	2.9994	0.0002	0.0001	3.0012	2.9973	0.0038	3.0003	0.0001	5.6687×10^{-5}	3.00069	2.9999	0.0007
	$\beta = 5$	4.9988	0.0002	0.0006	5.0004	4.99678	0.0037	4.9988	0.0002	0.0005	5.0000	4.9981	0.0018
50	$\alpha = 5$	4.9667	0.0066	0.4424	5.0262	4.9326	0.0935	4.9921	0.0015	0.0246	5.0202	4.9650	0.05520
	$\lambda = 3$	2.980	0.0065	0.1523	3.0744	2.8720	0.2024	3.0133	0.0044	0.0712	3.0362	2.9869	0.0493
	$\beta = 10$	9.9699	0.0030	0.3606	10.0109	9.9286	0.0822	10.0070	0.0007	0.0199	10.019	9.9934	0.0258
100	$\alpha = 5$	4.9600	0.0059	0.2381	4.9940	4.9137	0.0803	5.0040	0.0008	0.0067	5.0069	5.0008	0.0061
	$\lambda = 3$	2.9932	0.0026	0.0182	3.0168	2.9738	0.0429	3.0078	0.0022	0.0148	3.0170	2.9991	0.0179
	$\beta = 10$	10.0286	0.0028	0.3277	10.0523	9.9968	0.0555	9.9972	0.0002	0.0029	10.0027	9.9895	0.0131
200	$\alpha = 5$	5.0016	0.0003	0.0011	5.0061	4.9949	0.0111	5.0010	0.0002	0.0004	5.0045	4.9980	0.0065
	$\lambda = 3$	3.0060	0.0020	0.0144	3.0106	3.0012	0.00931	2.9977	0.0007	0.0020	2.9995	2.9962	0.0032
	$\beta = 10$	9.9828	0.0017	0.1181	9.9976	9.9758	0.0217	9.9988	0.0001	0.0005	10.0021	9.9945	0.0076

Table 4

Bayes predictors and bounds of the future observation

based on Type II censoring under two-sample prediction

$(N = 10000, n = 100,\ r\ =\ 80\%,, m = 25)$

s	SE				LINEX$(\boldsymbol{v} = \mathbf{0.5})$			
	$\widetilde{\boldsymbol{y}}_{(S)}$	UL	LL	Length	$\widetilde{\boldsymbol{y}}_{(S)}$	UL	LL	Length
1	0.9899	0.9985	0.9808	0.0177	1.0062	1.0120	1.0009	0.0111
10	6.9989	7.0055	6.9926	0.0128	7.0036	7.0074	7.0010	0.0064
20	8.9917	8.9976	8.9879	0.0097	8.9967	9.0001	8.9930	0.0071
25	15.0062	15.0166	14.9973	0.01922	14.9987	15.0021	14.9959	0.0061

Table 5

Bayes estimates and standard errors of the parameters for the real data sets based on Type-II censoring

<table>
<tr><th rowspan="2">Applications</th><th rowspan="2">$\boldsymbol{n}$</th><th rowspan="2">r</th><th rowspan="2">$\boldsymbol{\varphi}$</th><th colspan="2">SE</th><th colspan="2">LINEX$(\boldsymbol{v} = \mathbf{0.5})$</th></tr>
<tr><th>Estimates</th><th>standard errors</th><th>Estimates</th><th>standard errors</th></tr>
<tr><td rowspan="9">I</td><td rowspan="9">15</td><td rowspan="3">9</td><td>α</td><td>0.4993</td><td>0.0099</td><td>0.4992</td><td>0.0087</td></tr>
<tr><td>λ</td><td>0.4973</td><td>0.0129</td><td>0.5009</td><td>0.0075</td></tr>
<tr><td>β</td><td>3.0007</td><td>0.0130</td><td>2.9982</td><td>0.0101</td></tr>
<tr><td rowspan="3">12</td><td>α</td><td>0.5000</td><td>0.0092</td><td>0.5000</td><td>0.0047</td></tr>
<tr><td>λ</td><td>0.4994</td><td>0.0080</td><td>0.4995</td><td>0.0052</td></tr>
<tr><td>β</td><td>3.0015</td><td>0.0088</td><td>2.9990</td><td>0.0055</td></tr>
<tr><td rowspan="3">15</td><td>α</td><td>0.5017</td><td>0.0073</td><td>0.5000</td><td>0.0031</td></tr>
<tr><td>λ</td><td>0.5000</td><td>0.0062</td><td>0.4998</td><td>0.0043</td></tr>
<tr><td>β</td><td>3.0006</td><td>0.0059</td><td>2.9999</td><td>0.0037</td></tr>
<tr><td rowspan="9">I I</td><td rowspan="9">21</td><td rowspan="3">13</td><td>α</td><td>3.0028</td><td>0.0124</td><td>2.9988</td><td>0.0071</td></tr>
<tr><td>λ</td><td>2.0007</td><td>0.0078</td><td>2.0014</td><td>0.0061</td></tr>
<tr><td>β</td><td>13.0010</td><td>0.0086</td><td>13.0005</td><td>0.0074</td></tr>
<tr><td rowspan="3">17</td><td>α</td><td>2.9989</td><td>0.0061</td><td>3.0000</td><td>0.0053</td></tr>
<tr><td>λ</td><td>1.9995</td><td>0.0043</td><td>2.0010</td><td>0.0057</td></tr>
<tr><td>β</td><td>12.9998</td><td>0.0071</td><td>13.0006</td><td>0.0040</td></tr>
<tr><td rowspan="3">21</td><td>α</td><td>2.9984</td><td>0.0057</td><td>2.9998</td><td>0.0033</td></tr>
<tr><td>λ</td><td>1.9997</td><td>0.0044</td><td>1.9994</td><td>0.0037</td></tr>
<tr><td>β</td><td>12.9985</td><td>0.0053</td><td>13.0002</td><td>0.0028</td></tr>
</table>

Table 6

Bayes estimates and standard errors of the sf, hrf and ahrf for the real data sets based on Type-II censoring

Applications	n	r	φ	SE Estimates	SE standard errors	LINEX($v=0.5$) Estimates	LINEX($v=0.5$) standard errors
I	15	9	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8989 0.0722 0.0739	0.0115 0.0126 0.0113	0.8987 0.0699 0.0743	0.0076 0.0080 0.0078
		12	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8967 0.0732 0.0745	0.0074 0.0092 0.0076	0.8982 0.0712 0.0737	0.0054 0.0050 0.0047
		15	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8973 0.0734 0.0732	0.0068 0.0078 0.0051	0.8981 0.0719 0.0735	0.0037 0.0041 0.0040
I I	21	13	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8303 0.0621 0.0657	0.0149 0.0114 0.0123	0.8332 0.0648 0.0683	0.0059 0.0080 0.0054
		17	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8364 0.0659 0.0702	0.0097 0.0065 0.0094	0.8352 0.0657 0.0688	0.0050 0.0051 0.0053
		21	$S(x_0)$ $h(x_0)$ $ah(x_0)$	0.8326 0.0661 0.0687	0.0058 0.0042 0.0041	0.8337 0.0666 0.0687	0.0050 0.0043 0.0042

Table 7

Bayes predictors and bounds of the future observation based on Type II censoring under two-sample prediction for two real data sets

Applications	s	SE $\tilde{y}_{(S)}$	SE UL	SE LL	SE Length	LINEX($v=0.5$) $\tilde{y}_{(S)}$	LINEX UL	LINEX LL	LINEX Length
I	1	1.0033	1.0094	0.9975	0.0119	1.0025	1.0042	0.9997	0.0044
	5	2.9932	2.9988	2.9891	0.0096	2.9976	2.9994	2.9964	0.0029
	10	6.9973	7.0019	6.9922	0.0097	7.0008	7.0019	6.9994	0.0025
II	1	1.0066	1.0124	0.9993	0.0130	0.9942	0.9985	0.9910	0.0074
	8	4.9999	5.0027	4.9959	0.0068	5.0030	5.0052	5.0000	0.0052
	15	9.9986	10.0015	9.9945	0.0069	10.0033	10.0057	10.0007	0.0050