A Discrete Analog of the Alpha Power Inverted Kumaraswamy Distribution with Applications to Real-life Data Sets

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Abstract

In this paper, discrete alpha power inverted Kumaraswamy distribution is proposed. The general approach of discretization of a continuous distribution is used to derive discrete alpha power inverted Kumaraswamy distribution. Its probability mass function has different shapes including decreasing, increasing, upside-down bathtub and unimodal. The proposed distribution has three parameters and its hazard rate function has several shapes. Moreover, the proposed distribution can be used to analyze over and under dispersed count data. Some properties of the proposed distribution are studied including the quantiles, mean residual life, mean time between failures and mean time to failure, Rényi entropy, non-central moments, central moments, standard moments and order statistics. Maximum likelihood estimation is considered under Type-II censored samples for estimating the model parameters, survival, hazard rate and alternative hazard rate functions. Also, confidence intervals are constructed. Monte Carlo simulation is applied to demonstrate the theoretical results of the maximum likelihood estimates and confidence intervals.

Finally, two real data sets are presented to examine the precision of the maximum likelihood estimates and to ensure the simulated results.

Keywords: *Alpha power transformation; Inverted Kumaraswamy distribution; Discrete lifetime models; Survival and hazard rate functions; Order statistics; Maximum likelihood estimation.*

1. Introduction

Recently, some authors generalized classical distributions to extend their form to be more flexible for modeling real data. In the literature, several methods of generating new statistical distributions were presented; for example, Marshall and Olkin (1997), Eugene *et al.* (2002), Cordeiro and Castro (2011) and Alzaatreh *et al.* (2013). For more details about methods of generating distributions see, Lee *et al.* (2013) and Jones (2015).

Mahdavi and Kundu (2016) added an extra parameter to a family of distributions functions to let the given family more flexible. They called the new method *α-power transformation* (APT) method, which can be used quite effectively for purposes of data analysis. They proposed α-power exponential distribution which has desirable properties; such as the *cumulative distribution function* (cdf) is appropriate for analyzing censored data since it can be written in explicit form. Also, the *probability density function* (pdf) and *hazard rate function* (hrf) of α-power exponential distribution acts like Weibull, Gamma or generalized exponential distributions.

The APT of the cdf, $F(x)$, which is the cdf of a continuous random variable X, for $X \in \mathbb{R}$, is defined as follows:

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}, & \alpha > 0, \alpha \neq 1, \\ F(x), & \alpha = 1, \end{cases} \tag{1}$$

and the corresponding pdf is

$$f_{APT}(x) = \begin{cases} \frac{\ln(\alpha)}{\alpha-1} f(x)\alpha^{F(x)}, & \alpha > 0, \alpha \neq 1, \\ f(x), & \alpha = 1, \end{cases} \tag{2}$$

where α is a shape parameter.

Many researchers applied the APT method to many distributions, such as Nassar *et al.* (2017) presented the alpha power Weibull distribution. Dey *et al.* (2017) introduced the alpha power generalized exponential distribution. Nadarajah and Okorie (2018) studied the moment properties of the alpha power generalized exponential distribution. Mead *et al.* (2019) obtained some statistical properties of the APT family and considered the alpha power exponentiated Weibull distribution. Also, Nassar *et al.* (2020) discussed the parameter estimation of the alpha power exponential distribution using nine methods of estimation.

Although it is common in reliability lifetime modeling, to deal with failure data as continuous, indicating some degree of accuracy in measurement, practically; failures are observed at fixed inspection intervals, happened in a discrete procedure or are simply recorded in boxes. In survival analysis, the *survival function* (sf) may be a function of discrete random variable which is a discrete version of base continuous random variable. Such as, the survival time of patients is counted by number of days or weeks or the length of stay in an observation ward is counted by number of days. Also, in real life, the reliability data are measured in terms of the numbers of runs, cycles or shocks the device sustains before it fails. For example, the number of times the devices are switched on/off, the lifetime of the switch is a discrete random variable. Also, the number of voltage fluctuations; which an electrical or electronic item can withstand before its failure, the life of equipment is measured by the number of completed cycles or the number of times it operated before failure, or the life of weapon is measured by the number of rounds fired prior to failure.

Some known discrete distributions such as geometric, Poisson, binomial, beta binomial, multinomial, hypergeometric, negative binomial and etc., have limited applicability as models for failure times, reliability and counts. Therefore, it is realistic and suitable to model the discrete failure time by an appropriate discrete lifetime distribution generated from the base continuous distribution keeping one or more important characters of the continuous distribution. [For more details see, Lai (2013) and Chakraborty and Chakravorty (2016)].

There are different methods by which discrete correspondent random variable of a continuous random variable may be derived. [see, Bracquemond and Gaudoin (2003) and Chakraborty (2015)]. The general approach of discretization of some known continuous distributions have been attracting great interest for use as lifetime distributions by many researchers [see, Nakagawa and Osaki (1975), Khan *et al.* (1989), Inusah and Kozubowski (2006), Krishna and Pundir (2009), Jazi *et al.* (2010), Gomez-Deniz and Calderin-Ojeda (2011) and Nekoukhou *et al*. (2012)].

The problem of *Maximum likelihood* (ML) and Bayesian estimation for discrete distributions were considered by Al-Huniti and AL-Dayian (2012), Migdadi (2014), Kamari *et al.* (2015), Hegazy *et al*. (2018) and EL-Helbawy *et al*. (2022). Recently, many authors studied several discrete distributions [see, Lekshmi and Sebastian (2014), Para and Jan (2014), Hussain and Ahmad (2014), Hussain *et al*. (2016), Alamatsaz *et al*. (2016), Para and Jan (2016), Sarhan (2017) and Borah and Hazarika (2017)].

The general approach of discretizing a continuous variable can be used to construct a discrete model by introducing a grouping on the time axis [see, Roy (2003, 2004)]. If the continuous random variable X has the sf, $S(x) = P(X \geq x)$ and times are grouped into unit intervals so that the discrete random variable of X denoted $\lfloor X \rfloor$; which is the largest integer less than or equal to X, will have the *probability mass function* (pmf)

$$P(dX = x) = P(x) = P[x \leq X \leq x + 1]$$
$$= S(x) - S(x + 1), \qquad x = 0, 1, 2, \dots \quad (3)$$

One of the advantages of applying the general approach of discretizing is that the sf for discrete distributions has the similar functional form of the sf for the continuous distributions; hence, many reliability characteristics and properties keep on unchanged. Thus, discretization of a continuous lifetime model according to this approach is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one.

Abd AL-Fattah *et al.* (2017) introduced the *inverted Kumaraswamy* (IKum) distribution. This distribution is important in a wide range of applications; for example engineering, medical research and lifetime problems. It is effective in analyzing many lifetime data since it has failure rates that take decreasing and upside-down bathtub shapes depending on the value of the shape parameters. They observed that the exponential distribution, generalized exponential distribution, Weibull distribution, beta distribution, gamma distribution, uniform distribution, exponentiated exponential distribution, exponentiated Gamma distribution and other distributions can be obtained as special cases of the IKum distribution.

Considering X is a random variable from IKum distribution, then the pdf and cdf are, respectively, given by

$$f(t;\alpha,\beta) = \lambda\beta(1+x)^{-(\lambda+1)}\left(1-(1+x)^{-\lambda}\right)^{\beta-1}, \quad x>0;\ \lambda,\beta>0, \tag{4}$$

and

$$F(t;\alpha,\beta) = \left(1-(1+x)^{-\lambda}\right)^{\beta}, \quad x>0;\ \lambda,\beta>0, \tag{5}$$

where λ and β are the shape parameters.

Hozaien *et al*. (2020) presented the *alpha power inverted Kumaraswamy* (APIKum) distribution and studied some of its statistical properties.

$$g_1(x;\alpha,\lambda,\beta) = \frac{ln(\alpha)}{\alpha-1}\lambda\beta(1+x)^{-(\lambda+1)} \times (1-\left(1+x)^{-\lambda}\right)^{\beta-1}\alpha^{(1-\left(1+x)^{-\lambda}\right)^{\beta}}, \quad x>0, \alpha,\lambda,\beta>0,\ \alpha\neq 1 \tag{6}$$

and

$$G_1(x;\alpha,\lambda,\beta) = \frac{\alpha^{(1-\left(1+x)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}, \quad x>0,\ \alpha,\lambda,\beta>0, \alpha\neq 1. \tag{7}$$

The sf; $S_1(x)$, and the hrf; $h_1(x)$, are respectively, given by

$$S_1(x;\alpha,\lambda,\beta) = \frac{\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1}, \quad \alpha\neq 1, \tag{8}$$

and

$$h_1(x;\alpha,\lambda,\beta) = \frac{ln(\alpha)\,\lambda\beta(1+x)^{-(\lambda+1)}\left(1-\left(1+x\right)^{-\lambda}\right)^{\beta-1}\,\alpha^{\left(1-\left(1+x\right)^{-\lambda}\right)^{\beta}}}{\alpha-\alpha^{\left(1-\left(1+x\right)^{-\lambda}\right)^{\beta}}}, \quad \alpha\neq 1. \tag{9}$$

The paper is organized as follows: Constructing discrete alpha power inverted Kumaraswamy distribution is presented in Section 2. Some statistical properties are obtained in Section 3. Maximum likelihood estimation for the unknown parameters, survival function, hazard rate and alternative hazard rate functions are derived, under Type-II censored samples, in Section 4. Finally in Section 5, a numerical illustration is introduced.

2. Discrete Alpha Power Inverted Kumaraswamy Distribution

EL-Helbawy *et al*. (2022) derived *discrete inverted Kumaraswamy* (DIKum) distribution using the general approach of discretization of a continuous distribution. They obtained some important distributional, reliability properties and ML estimators for the DIKum distribution. From (3) *discrete X* (*dX)* can be viewed as the discrete analogue to the continuous APIKum variable *X*, and is commonly said to follow APIKum distribution, named *Discrete alpha power inverted Kumaraswamy* (DAIKum) distribution and is denoted by DAPIKum (α,λ,β) distribution. Using (3) and (8) the corresponding pmf of DAIKum distribution can be written as

$$P(x) \equiv P(x;\alpha,\lambda,\beta) = S(x) - S(x+1)$$

$$= \frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - 1},$$

$$x = 0, 1,2, \dots, \quad \alpha,\lambda,\beta > 0, \alpha \neq 1 \qquad (10)$$

and the cdf, sf and hrf are as follows:

$$F(x) \equiv F(x;\alpha,\lambda,\beta) = P(X \leq x)$$

$$= 1 - S(x) + P(X = x)$$

$$= \frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}} - 1}{\alpha - 1}, \ \alpha \neq 1, \qquad (11)$$

$$S(x) \equiv S(x;\alpha,\lambda,\beta) = P(X \geq x)$$

$$= 1 - F(x) + P(X = x),$$

$$= \frac{\alpha - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - 1},$$

$$x = 0, 1, 2, \dots, \quad \alpha \neq 1, \qquad (12)$$

and

$$h(x) \equiv h(x;\alpha,\lambda,\beta) = \frac{P(x)}{S(x)}$$

$$= \frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}. \qquad (13)$$

There are some problems associated with the definition of $h(x)$, three of the more notable ones are:

a. $h(x)$ is not additive for series system.
b. The cumulative hrf, $H(x) = \sum h(x) \neq -\ln S(x)$.
c. $h(x) \leq 1$ and it has the interpretation of a probability.

Roy and Gupta (1992) provided an excellent alternative definition of a discrete hrf; *alternative* hrf (ahrf); $ah(x)$, since it was needed to find an alternative definition that is consistent with its continuous counterpart [see, Xie (2002) Lai (2013) and (2014)].

$$ah(x) \equiv ah(x;\alpha,\lambda,\beta) = ln\left[\frac{S(x)}{S(x+1)}\right]$$

$$= ln\left[\frac{\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-\alpha^{(1-(2+x)^{-\lambda})^{\beta}}}\right] \ ,$$

$$x = 0,1,2,\dots\ ,\ \alpha,\lambda,\beta > 0, \alpha \neq 1\ . \qquad (14)$$

The two concepts $h(x)$ and $ah(x)$ have the same monotonic property, i.e., $ah(x)$ is increasing (decreasing) if and only if $h(x)$ is increasing (decreasing).

Plots of pmf, hrf and ahrf of DAPIKum (α,λ,β) distribution are presented, respectively, in Figures 1-3, for some selected values of the parameters.

The *reversed hazard rate* (rhr) function is

$$rh(x) = \frac{P(x)}{F(x)}$$

$$= \frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}}-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha^{(1-\left(2+x\right)^{-\lambda})^{\beta}}-1},$$

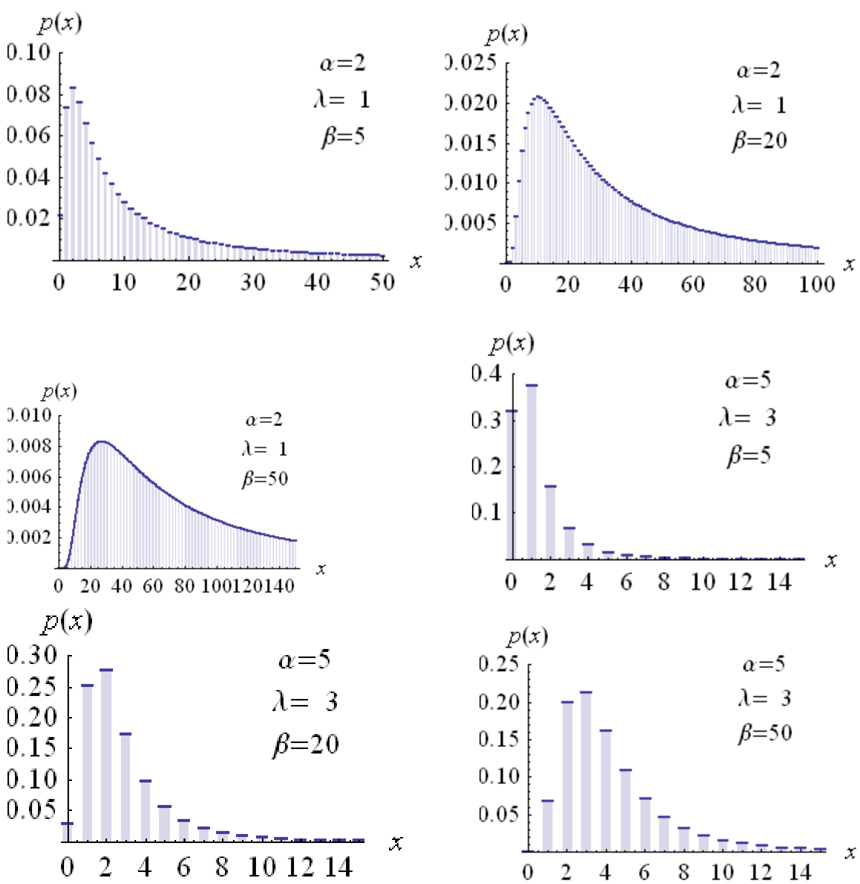
$$x = 0,1,2,\dots, \quad \alpha,\lambda,\beta > 0, \quad \alpha \neq 1\ . \qquad (15)$$

Figure 1, displays the pmf of DAIKum distribution for different values of parameters. Figure 2 and 3 show, respectively, the hrf of DAIKum distribution and the ahrf for different values of the parameters.

From the plots in Figures 1-3, one can observe that the pmf of DAIKum distribution can be increasing, decreasing, upside-down bathtub and unimodal and right skewed according to the chosen values of the parameters. For some values of parameters, the pmf is decreasing over $(0,\infty)$ and the mode is at zero. While for other values of the parameters, it indicates that the pmf is increasing on $(0,x_{mode})$ and reaches the maximum at x_{mode}, then decreases to the zero on (x_{mode},∞). Plots of pmf, hrf and ahrf show that the

DAIKum distribution shows a long right tail compared with other regularly used distributions. Thus, it can provide a good fit to several data in literature since it will affect long term reliability predictions, producing optimistic predictions of rare events arising in the right tail of the distribution compared with other distributions.

Figures of hrf and ahrf indicate that although the hrf and ahrf of DAIKum distribution have several shapes depending on the value of the parameters, the hrf is less than 1.



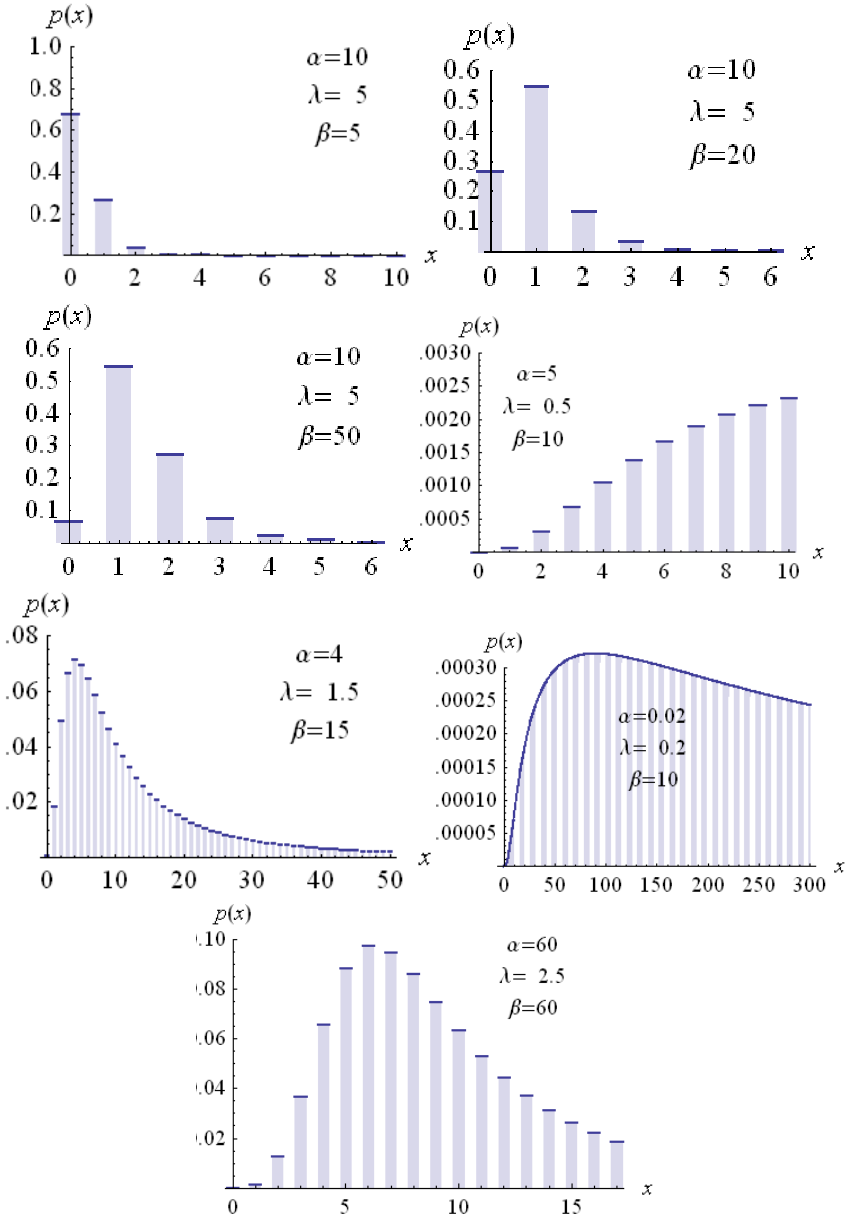
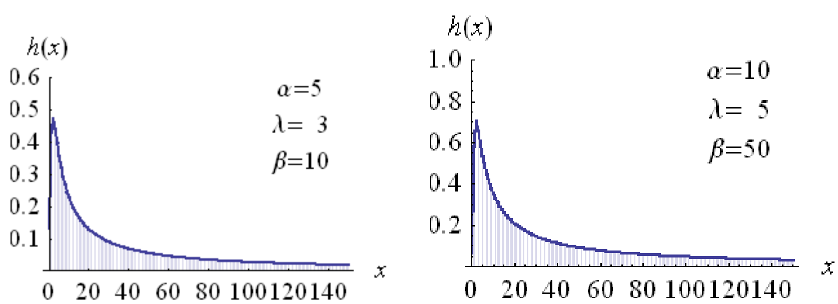


Figure1. Plots of the probability mass function of DAIKum



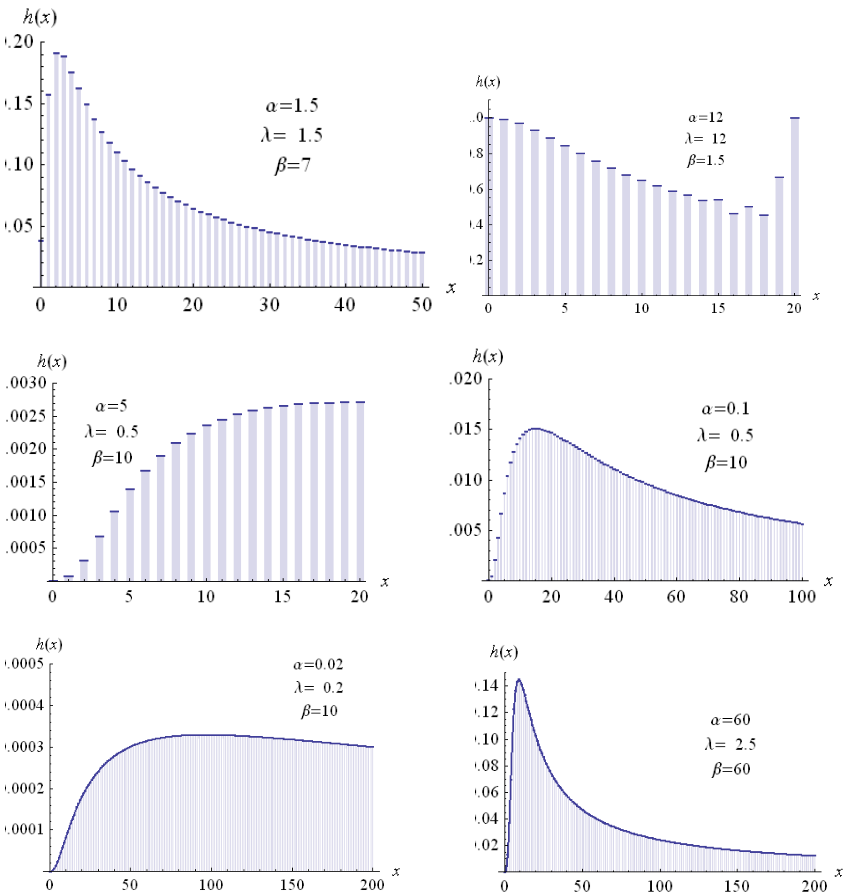
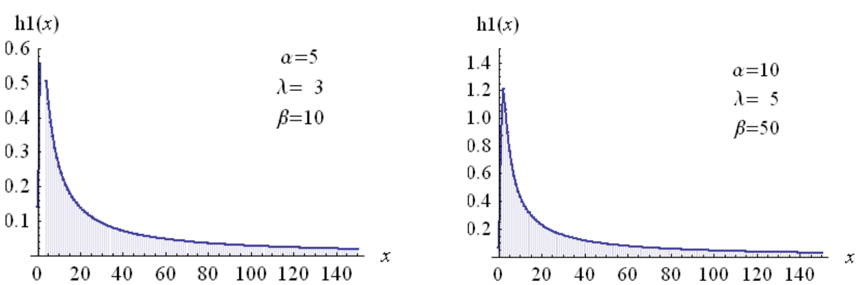


Figure2. Plots of the hazard rate function of DAIKum



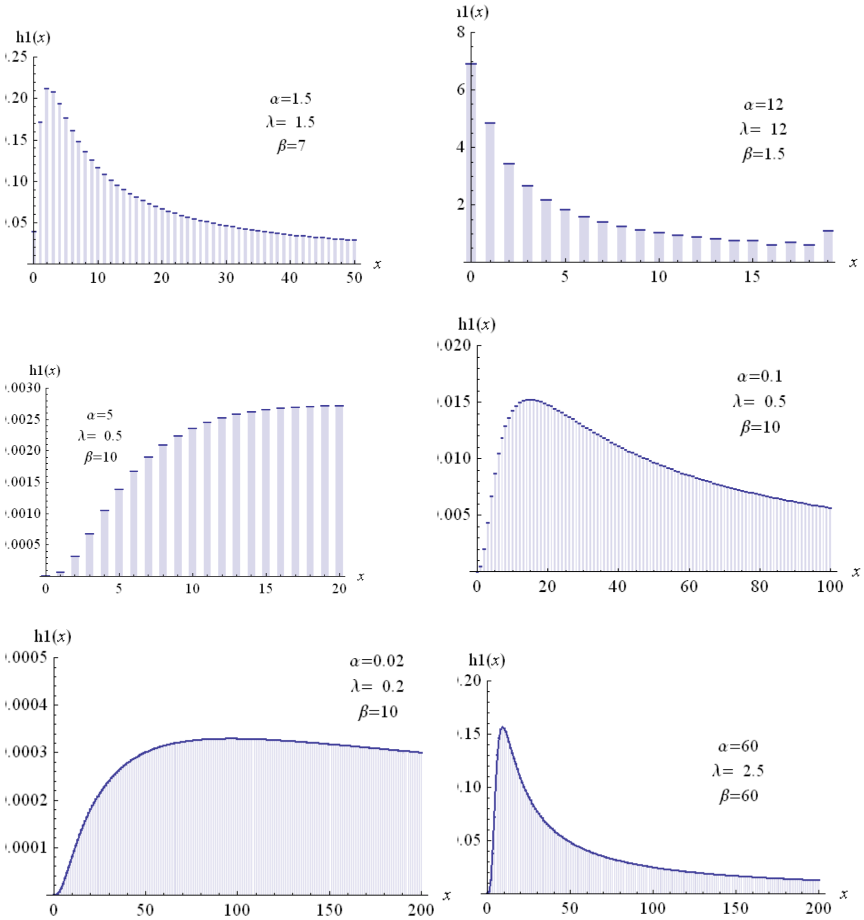


Figure3. Plots of the alternative hazard rate function of DAIKum

3. Some Properties of Discrete Alpha Power Inverted Kumaraswamy Distribution

This section is dedicated to obtain some important distributional properties of DAPIKum (α, λ, β) distribution, such as the quantiles, r^{th} moments and order statistics.

3.1 Quantiles of discrete alpha power inverted Kumaraswamy distribution

The u^{th} quantile of a discrete random variable$X$; x_u, satisfies

$P(X \leq x_u) \geq u$ and $P(X \geq x_u) \geq 1-u$, i.e.,

$F(x_u - 1) < u \leq F(x_u)$. [For more details see, Rohatgi and Saleh (2001)].

Hence

The u^{th}quantile x_u, of the DAPIKum $(\alpha, \beta, \lambda\ )$ distribution is given by

$$x_u = \left\lceil \left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{1/\beta}\right]^{-1/\lambda} - 2 \right\rceil, 0 < u < 1, \qquad (16)$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and $0 < u < 1$.

Proof

$P(X \leq x_u) \geq u$, from (11)

$$\frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - 1}{\alpha - 1} \geq u\ ,$$

then

$$x_u \geq \left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{-\frac{1}{\lambda}} - 2\,. \qquad (17)$$

Similarly, if $P(X \geq x_u) \geq 1-u$, one obtains

$$x_u \leq \left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{-\frac{1}{\lambda}} - 1. \qquad (18)$$

Combining (16) and (17), one gets,

$$\left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{-\frac{1}{\lambda}} - 2 \leq x_u \leq \left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{-\frac{1}{\lambda}} - 1, \qquad (19)$$

Hence, x_u is an integer value given by

$$x_u = \left\lceil \left[1 - \left[\frac{ln[(\alpha-1)u+1]}{ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{-\frac{1}{\lambda}} - 2 \right\rceil\ . \qquad (20)$$

Thus, the median of DAPIKum (α, λ, β) distribution can be computed from (20) as follows:

$$x_{0.5} = \left[\left[1 - \left[\frac{ln[(\alpha-1)(0.5)+1]}{ln(\alpha)} \right]^{\frac{1}{\beta}} \right]^{-\frac{1}{\lambda}} - 2 \right]. \quad (21)$$

3.2 The moments of the discrete alpha power inverted Kumaraswamy distribution

a. The non-central moments of the discrete alpha power inverted Kumaraswamy distribution

The non-central moments of DAPIKum distribution can be obtained using (10) as follows:

$$\mu'_r = E(X^r) = \sum_{x=0}^{\infty} x^r P(x) \quad (22)$$

$$= \sum_{x=0}^{\infty} x^r \left(\frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1} \right), \quad r = 1, 2, \ldots,$$

$$\alpha \neq 1 . \quad (23)$$

In particular, the mean (μ) of DAIKum distribution is given by

$$\acute{\mu}_1 = \mu = \sum_{x=0}^{\infty} x \left(\frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1} \right) . \quad (24)$$

b. The central moments of the discrete alpha power inverted Kumaraswamy distribution

The central moments can be derived using the relation between the central and non-central moments as given below

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1^j) \mu^j \acute{\mu}_{r-j} , \qquad r = 1,2, \ldots., \quad (25)$$

thus, the variance (μ_2) of DAIKum distribution is

$$\mu_2 = \sum_{x=0}^{\infty} X^2 \left(\frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1} \right) - \left[\sum_{x=0}^{\infty} x \left(\frac{\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1} \right) \right]^2 . \quad (26)$$

c. The index of dispersion of the discrete alpha power inverted Kumaraswamy distribution

The *index of dispersion* (ID) is defined as the ratio between the variance to the mean. The ID indicates that the distribution is suitable for under-, equi- or over-dispersed data sets. If the $ID > 1$ (< 1) the distribution is over dispersed (under dispersed) and if $\text{ID} = 1$ the distribution is equi-dispersed.

d. The standard moments of the discrete alpha power inverted Kumaraswamy distribution

The r^{th} standard moments can be obtained as follows:

$$\alpha_r = E\left(\frac{X-\mu}{\sigma}\right)^r. \qquad (27)$$

The skewness and kurtosis of the DAIKum distribution are given, respectively, by

$\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$, where

$$\mu_r = \text{E}(X-\mu)^r, \quad r = 1,2, \dots .$$

Table 1 presents these results numerically for different values of the parameters. From Table 1, one can observe that depending on the values of the parameters, the variance of the distribution can be smaller or greater than the mean. Hence DAIKum distribution is appropriate for modeling both over and under dispersed data. The skewness of DAIKum distribution is always positive which means that the distribution is right skewed. The kurtosis for most values is greater than 3 which mean that the DAIKum is heavy tail.

Table1
Mean, median, variance, index of dispersion, skewness and kurtosis of DAIKum distribution for some values of α, λ and β

Parameters	Mean	Median	Variance	Index of Dispersion	Skewness	Kurtosis
$\alpha = 2$ $\lambda = 1$ $\beta = 5$	9.4761	8	106.3020	11.2179	2.0124	6.3611
$\alpha = 2$ $\lambda = 1$ $\beta = 10$	12.2754	18	124.4680	10.1396	1.9169	5.2648
$\alpha = 2$ $\lambda = 1$ $\beta = 20$	13.1657	36	140.4870	10.6706	2.0909	5.1080
$\alpha = 5$ $\lambda = 1$ $\beta = 5$	10.8016	12	119.0020	11.0170	1.8997	5.5803
$\alpha = 5$ $\lambda = 1$ $\beta = 10$	12.7013	25	133.8940	10.5417	1.976	5.1212
$\alpha = 5$ $\lambda = 1$ $\beta = 20$	11.9242	51	277.265	19.0591	2.0951	5.1500
$\alpha = 2$ $\lambda = 3$ $\beta = 5$	1.3588	1	3.5581	2.6185	5.3818	65.4632
$\alpha = 2$ $\lambda = 3$ $\beta = 20$	2.5256	2	6.4464	2.5524	4.6676	44.8514
$\alpha = 5$ $\lambda = 5$ $\beta = 20$	0.8746	1	0.7538	0.8619	2.3942	24.3036
$\alpha = 10$ $\lambda = 5$ $\beta = 5$	0.3895	0	0.4505	1.1566	2.9474	28.4605
$\alpha = 10$ $\lambda = 5$ $\beta = 20$	0.9980	1	0.8096	0.8112	2.4559	25.159
$\alpha = 10$ $\lambda = 5$ $\beta = 50$	1.5099	1	1.0714	0.7095	2.8434	28.0426

3.3 The order statistics of the discrete alpha power inverted Kumaraswamy distribution

Let $F_i(x;\alpha,\lambda,\beta)$; the cdf of the i^{th} order statistic for a random sample$X_1, X_2, \dots, X_n$, from the DAPIKum (α,λ,β), is given by

$$F_i(x;\alpha,\lambda,\beta) = \sum_{r=i}^{n} \binom{n}{r} [F(x;\alpha,\lambda,\beta)]^r [1-F(x;\alpha,\lambda,\beta)]^{n-r}. \tag{28}$$

Using the binomial expansion for $[1-F_i(x;\alpha,\lambda,\beta)]^{n-r}$ and substituting (11) in (28), one gets

$$F_i(x;\alpha,\lambda,\beta) = \sum_{r=i}^{n} \binom{n}{r} [F(x;\alpha,\lambda,\beta)]^r \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x;\alpha,\lambda,\beta)]^j .$$

$$= \sum_{r=i}^{n} \binom{n}{r} \left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^r \times \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^j$$

$$= \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n} \binom{n-r}{j} (-1)^j \left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{r+j} . \tag{29}$$

Special cases

Case I: If $i = 1$ in (29) one can obtain the distribution function of the first order statistic, as given below

$$F_1(x;\alpha,\lambda,\beta) = 1-[1-F(x;\alpha,\lambda,\beta)]^n$$

$$= 1-\left[1-\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^n . \tag{30}$$

Case II: If $i = 1$ in (29) the distribution function of the largest order statistic, as follows:

$$F_n(x;\alpha,\lambda,\beta) = [F(x;\alpha,\beta)]^n$$

$$= \left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^n. \qquad (31)$$

Suppose that $X_1, X_2, \dots, X_n$is a random sample from the DAPIKum (α,λ,β) distribution.

Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics.

Hence, the pmf of $X_{i:n}$, is defined by

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\int_{F(x-1)}^{F(x)} v^{i-1}(1-v)^{n-i}dv. \qquad (32)$$

Using the binomial expansion for $(1-v)^{n-i}$, then, the pmf in (32) is

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\int_{F(x-1)}^{F(x)} v^{i+j-1}dv$$

$$= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j \left(\frac{1}{i+j}\right)$$

$$\times\left\{\left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{i+j} - \left[\frac{\alpha^{\left(1-\left(1+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{i+j}\right\}. \qquad (33)$$

The pmf of the smallest order statistic is obtained by substituting $i = 1$ in (33) as follows:

$$P(X_{1:n} = x) = n\sum_{j=0}^{n-i}\binom{n-1}{j}(-1)^j \left(\frac{1}{1+j}\right)$$

$$\times\left\{\left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{1+j} - \left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{1+j}\right\}, \qquad (34)$$

and the pmf of the largest order statistic is obtained by substituting $i = n$ in (33) as follows:

$$P(X_{n:n}=x)= n\sum_{j=0}^{n-i}\binom{n-1}{j}(-1)^j\left(\frac{1}{1+j}\right)$$
$$\times\left\{\left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{n+j}-\left[\frac{\alpha^{\left(1-\left(2+x\right)^{-\lambda}\right)^{\beta}}-1}{\alpha-1}\right]^{n+j}\right\}. \qquad (35)$$

Also, (29) can be used to obtain the pmf of the DAPKum (α,λ,β) distribution, (see Arnold *et al*. (2008)).

3.4 Mean residual life of the discrete alpha power inverted Kumaraswamy distribution

The m*ean residual life* (MRL) is the expected remaining life, $X-x_0$, given that the item has survived to time x_0 [see, Kemp (2004)] it is denoted by $m(x_0)$ and is defined by

$$m(x_0)=\frac{\sum_{k=x_0+1}^{\infty}S(k)}{S(x_0)}$$
$$=\frac{\alpha-\sum_{k=x_0+1}^{\infty}\alpha^{(1-(1+k)^{-\lambda})^{\beta}}}{\alpha-\alpha^{(1-(1+x_0)^{-\lambda})^{\beta}}}\ . \qquad (36)$$

3.5 Mean time between failures and mean time to failure of the discrete alpha power inverted Kumaraswamy distribution

Mean Time to Failure (MTTF) is the average time between non-repairable failures and is generally used for items that cannot be repaired, such a light bulb or a backup tape. The average time a device or system is expected to function before it fails. Typically, information technology teams collect this data by observing system components for several days or weeks. While like *Mean Time between Failure* (MTBF), MTTF is normally used to describe replaceable items such as a tape drive in a backup array, whereas MTBF is used with items that can be either repaired or replaced.

It predicts the failure rate for products that cannot be repaired. It's important for organizations to be aware of the difference between these three concepts, so they don't waste time focusing on how long it takes to repair a system when the best option could be to replace it with a new one.

The MTBF is given by

$$MTBF = \frac{-x}{log[S(x)]}$$

$$= \frac{-x}{log[\frac{\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1}]}\ , \quad x > 0,\ \alpha \neq 1. \tag{37}$$

The MTTF is given by

$$MTTF = \sum_{x=0}^{\infty} S(x)$$

$$= \sum_{x=0}^{\infty} (\alpha-1)^{-1}\left(\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}\right)\ ,\ \ x > 0,\ \alpha \neq 1. \tag{38}$$

The *Availability* (Av) is considered as being the probability that the component is successful at time x, i.e.,

$$Av = \frac{MTTF}{MTBF}$$

$$= \frac{\left[\sum_{x=0}^{\infty}(\alpha-1)^{-1}\left(\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}\right)\right]\left[log[\frac{\alpha-\alpha^{(1-(1+x)^{-\lambda})^{\beta}}}{\alpha-1}]\right]}{-x}. \tag{39}$$

3.6 R*é*nyi entropy of the discrete alpha power inverted Kumaraswamy distribution

An entropy of a random variable X with the pdf $P(x)$ is a measure of variation of the uncertainty and it is denoted by $H_R(\rho)$. It has been applied in a wide variety of fields such as statistical thermodynamics, urban and regional planning, business, economics, finance, operations research, queueing theory, spectral analysis, image reconstruction, biology and manufacturing. It is defined by

$$H_R(\rho) = (1-\rho)^{-1} log\{\sum_{x=0}^{\infty}(P(x))^{\rho}\}$$

$$= \left((1-\rho)(\alpha-1)\right)^{-1} \sum_{x=0}^{\infty}\left(\alpha^{(1-(2+x)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x)^{-\lambda})^{\beta}}\right)^{\rho},$$

$$\alpha \neq 1\ \rho > 0, \rho \neq 1\,. \quad (40)$$

The Shannon entropy can be defined by $E[-log(P(X))]$, and it can be calculated as a special case of the Rényi entropy when $\rho \to 1$.

4. Maximum Likelihood Estimation

This section is devoted to estimate the vector of parameters, $\underline{\varphi} = (\alpha, \lambda, \beta)$, sf, hrf and ahrf of the DAPIKum (α, λ, β) distribution, based on Type II censored samples, also confidence interval of the parameters $\alpha, \lambda, \beta,$ sf, hrf and ahrf are derived.

Suppose that $X_1, X_2, \ldots, X_r$ is a Type II censored sample of size *r* obtained from a life-test on n items whose lifetimes have a DAPKum (α, λ, β) distribution. Then the likelihood function is

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \{\prod_{i=1}^{r} P\left(x_{(i)}\right)\}\left[S\left(x_{(r)}\right)\right]^{n-r}, \quad (41)$$

where $P(x)$ and $S(x)$are given, respectively, by (10) and (12).

The $x_{(i)}$'s are ordered times for $i = 1,2, \ldots r$.

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} \frac{\alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}}}{\alpha-1}\right\}$$

$$\times \left\{\frac{\alpha-\alpha^{(1-\left(1+x_r\right)^{-\lambda})^{\beta}}}{\alpha-1}\right\}^{n-r}. \quad (42)$$

The ML estimators for α, λ and β are obtained by maximizing the natural logarithm of the likelihood function, denoted by ℓ which can be written in the form:

$$\ell \equiv ln\, L\left(\underline{\varphi}; \underline{x}\right) = ln \prod_{i=1}^{r} \frac{\alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}}}{\alpha - 1}$$

$$+(n-r) ln \left[\frac{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}{\alpha - 1}\right]. \qquad (43)$$

$$= \sum_{i=1}^{r} ln\left(\alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} - \alpha^{\left(1-(1+x_i)^{-\lambda})^{\beta}\right)} \right)$$

$$-r\, ln(\alpha - 1) + (n-r)\;\; ln\left(\alpha - \alpha^{(1-\left(1+x_r\right)^{-\lambda})^{\beta}} \right)$$

$$-(n-r)\, ln(\alpha - 1)$$

$$=\sum_{i=1}^{r} ln\left(\alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} - \alpha^{\left(1-(1+x_i)^{-\lambda})^{\beta}\right)} \right) - n\, ln(\alpha - 1)$$

$$+(\mathrm{n}-\mathrm{r}) ln\left(\alpha - \alpha^{(1-\left(1+x_r\right)^{-\lambda})^{\beta}} \right). \qquad (44)$$

Differentiating the log likelihood function in (44) with respect to α, λ and β, one obtains

$$\frac{\partial \ell}{\partial \alpha} = \frac{(1-(2+x_i)^{-\lambda})^{\beta} \alpha^{(1-\left(2+x_i\right)^{-\lambda})^{\beta}-1} - (1-(1+x_i)^{-\lambda})^{\beta} \alpha^{(1-\left(1+x_i\right)^{-\lambda})^{\beta}-1}}{\alpha^{(1-\left(1+x_i\right)^{-\lambda})^{\beta}} - \alpha^{(1-\left(2+x_i\right)^{-\lambda})^{\beta}}}$$

$$-\frac{n}{\alpha - 1} + (n-r)\,\frac{1-(1-(1+x_r)^{-\lambda})^{\beta} \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}-1}}{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}, \qquad (45)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{1}{\alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}}}$$

$$\times \Big[\beta(1-(2+x_i)^{-\lambda})^{\beta-1} \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} (2+x_i)^{-\lambda}$$

$$\times\, ln(2+x_i)\, ln(\alpha) - \beta(1-(1+x_i)^{-\lambda})^{\beta-1} \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} (1+x_i)^{-\lambda}$$

$$\times\, ln(1+x_i)\, ln(\alpha)]$$

$$-(n-r)\,\frac{\beta\left(1-(1+x_r)^{-\lambda}\right)^{\beta-1} \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}} (1+x_r)^{-\lambda}\, ln(1+x_r)\, ln\, \alpha}{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}$$

$$= \frac{\beta \ln(\alpha)}{\alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}}}$$

$$\times \Big[(1-(2+x_i)^{-\lambda})^{\beta-1} \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} (2+x_i)^{-\lambda}\, ln(2+x_i)$$

$$-(1-(1+x_i)^{-\lambda})^{\beta-1} \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} (1+x_i)^{-\lambda} \ln(1+x_i)\Big]$$

$$-(n-r)\frac{\beta \ln\alpha \left(1-(1+x_r)^{-\lambda}\right)^{\beta-1} \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}} (1+x_r)^{-\lambda} \ln(1+x_r)}{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}, \quad (46)$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{\Big\{ (1-(2+x_i)^{-\lambda})^{\beta} \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} \ln\left(1-(2+x_i)^{-\lambda}\right)}{\alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}}}$$

$$\times \ln(\alpha) - (1-(1+x_i)^{-\lambda})^{\beta} \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}}$$

$$\times \ln\left(1-(1+x_i)^{-\lambda}\right) \ln(\alpha)\Big\}$$

$$-(n-r)\frac{(1-(1+x_r)^{-\lambda})^{\beta} \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}} \ln(1-(1+x_r)^{-\lambda}) \ln(\alpha)}{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}$$

$$= \frac{\ln(\alpha)}{\alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} - \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}}}$$

$$\times \{(1-(2+x_i)^{-\lambda})^{\beta} \alpha^{(1-(2+x_i)^{-\lambda})^{\beta}} \ln\left(1-(2+x_i)^{-\lambda}\right)$$

$$-(1-(1+x_i)^{-\lambda})^{\beta} \alpha^{(1-(1+x_i)^{-\lambda})^{\beta}} \ln\left(1-(1+x_i)^{-\lambda}\right)\}$$

$$-(n-r)\frac{\left(1-(1+x_r)^{-\lambda}\right)^{\beta} \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}} \ln\left(1-(1+x_r)^{-\lambda}\right) \ln(\alpha)}{\alpha - \alpha^{(1-(1+x_r)^{-\lambda})^{\beta}}}. \quad (47)$$

The ML estimators can be obtained by equating the first partial derivatives of ℓ with respect to α, λ and β, respectively, to zeros. The system of non-linear equations (45)-(47) can be solved numerically using the Newton-Raphson method, to obtain the ML estimators $\hat{\alpha}, \hat{\lambda}$ and $\hat{\beta}$. The ML estimators $\hat{\alpha}, \hat{\lambda}$ and $\hat{\beta}$ have an asymptotic variance-covariance matrix defined by inverting the observed information matrix. Depending on the invariance property, the ML estimators of the sf, hrf and ahrf could be estimated by replacing α, β and λ with their corresponding ML estimators; $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ respectively, in (12), (13) and (14).

The asymptotic variance-covariance matrix for the estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are derived relying on $\tilde{I}^{-1}$ using the second derivatives of ℓ, where $\tilde{I}$ is the asymptotic Fisher information matrix and can be written as follows:

$$\tilde{I} \approx \left[\frac{\partial^2 \ell}{\partial \psi_i \psi_j}\right] \qquad i, j = 1,2,3, \tag{48}$$

where $\psi_1 = \alpha, \psi_3 = \lambda, and\ \psi_3 = \beta$.

- **The Asymptotic Confidence Intervals Based on Type II Censoring**

Considering that the size of the sample is large and under appropriate regularity conditions, then the ML estimators will be consistent, asymptotically unbiased and asymptotically normally distributed. Thus, the two sided approximate $100(1-\tau)\%$ confidence intervals for the parameters say ω can be obtained by $P\left(-z < \frac{\widehat{\omega} - \omega}{\sigma_{\widehat{\omega}}} < z\right) = 1 - \tau$, where z is the $100(1-\tau)$th standard normal percentile. The two sided approximate

$100(1-\tau)\%$ confidence intervals for ω, are:

$$L_\omega = \widehat{\omega} - z_{\frac{\sigma}{2}} \hat{\sigma}_{\widehat{\omega}} \qquad \text{and} \qquad U_\omega = \widehat{\omega} - z_{\frac{\sigma}{2}} \hat{\sigma}_{\widehat{\omega}}, \tag{49}$$

where $\hat{\sigma}_{\widehat{\omega}}$ is the standard deviation and $\widehat{\omega}$ is, respectively, $\hat{\alpha}, \hat{\beta}$, $\hat{\lambda}, \hat{S}(x), \hat{h}(x)$ or $\widehat{ah}(x)$.

Remark: All the results obtained under Type II censoring can reduce to the complete sample case when *r* = *n*.

5. Numerical Illustration

5.1 Simulation study

This subsection focuses on demonstrating the theoretical estimation results using simulated data.

5.1.1 Simulation algorithm

- A combination of the population parameter values for α, λ and β are used to generate several data sets from DAIKum distribution. Samples of size *n*=30, 60 and 120, are drawn from the population distribution under the complete sample case and when the data are censored at 60% and 80%, for each sample size where the *number of replications* (NR)=5000.
- The random samples are generated from the DAPIKum(α, β, λ) distribution using the following transformation:

$$x_i = \left\lceil \left[1 - \left[\frac{ln[(\alpha-1)u_i+1]}{ln(\alpha)}\right]^{1/\beta}\right]^{-1/\lambda} - 2 \right\rceil, i = 1, 2, \dots, n,$$

where u_i are random samples

from the uniform distribution (0,1), and then taking the ceiling.

- The nonlinear logarithmic likelihood equations can be solved simultaneously using Mathematica 9 applying the iterative technique of Newton Raphson method.
- The estimates $\hat{\alpha}, \hat{\lambda}$ *and* $\hat{\beta}$ are used to obtain the sf, hrf and ahrf estimates.
- Some measurements of accuracy are considered to evaluate the performance of the estimators,

$\hat{\alpha}$, $\hat{\lambda}, \hat{\beta}, \hat{S}(x), \hat{h}(x)$ and $\widehat{ah}(x)$. In order to study The variation and precision of the ML estimates is studied through the variance, the *relative absolute bias*

- $(RAB) = \frac{|estimate - population\ parameter|}{population\ parameter}$,

  and

  the relative error $(RE) = \frac{\sqrt{mean\ square\ error(estimate)}}{population\ paramater}$.

- Using (49), the asymptotic confidence intervals can be obtained for $\alpha, \lambda\ \beta$ sf, hrf and ahrf.
- The results are displayed in Tables 2-5.

5.2 Applications

In this subsection, two real data sets are provided to ensure the importance of the DAIKum distribution and to show how the proposed distribution can be used in real life. The estimates and their corresponding standard errors (SE) for the parameters, sf, hrf, ahrf for the two real data are shown in Table 6.

Application 1

The data of this application is considered by Mubarak and Almetwally (2021). This data represents COVID-19 data which belong to the United Kingdom of 76 days, from 15 April to 30 June 2020. These data formed of drought mortality rate. The data is: 0.0587 0.0863 0.1165 0.1247 0.1277 0.1303 0.1652 0.2079 0.2395 0.2751 0.2845 0.2992 0.3188 0.3317 0.3446 0.3553 0.3622 0.3926 0.3926 0.4110 0.4633 0.4690 0.4954 0.5139 0.5696 0.5837 0.6197 0.6365 0.7096 0.7193 0.7444 0.8590 1.0438 1.0602 1.1305 1.1468 1.1533 1.2260 1.2707 1.3423 1.4149 1.5709 1.6017 1.6083 1.6324 1.6998 1.8164 1.8392 1.8721 1.9844 2.1360 2.3987 2.4153 2.5225 2.7087 2.7946 3.3609 3.3715 3.7840 3.9042 4.1969 4.3451

4.4627 4.6477 5.3664 5.4500 5.7522 6.4241 7.0657 7.4456 8.2307 9.6315 10.1870 11.1429 11.2019 11.4584.

Application 2

The real data set of this application is obtained from Freireich *et al*. (1963). It represents the remission times (in weeks) for 21 patients who treated with placebo from 97 patients with acute leukemia participated in a clinical trial investigating the effect of 6-mercaptopurine. The remission times for the $n = 21$ patients treated with placebo were 1, 1, 2, 2, 3, 4, 4, 5, 5, 8, 8, 8, 8, 11, 11, 12, 12, 15, 17, 22, and 23 weeks.

Kolmogorov-Smirnov goodness of fit test is performed for each data set, to check the validity of the fitted model, and the p values are given, respectively, by 0.9021and 0.5860. In each case, the p value shows that the model fits the data very well. The results are displayed in Table 7 and 8.

The real data sets are provided to illustrate the flexibility and applicability of DAIKum distribution. DAIKum distribution is compared to some distributions such as *discrete inverse Weibull* (DIW) distribution introduced by Jazi *et al*. (2010), *generalization of discrete Weibull* (GDW) proposed by Jayakumar and Sankran (2017), *three parameter discrete generalized inverse Weibull* (DGIW) presented by Para and Jan (2018), *discrete Marshall-Olkin Weibull* (DMOW) derived by Opone and Akata (2021), *discrete Marshall-Olkin generalized exponential* (DMOGE) obtained by Almetwally *et al.* (2020), discrete modified inverse Rayleigh (DMIR) distribution proposed by Shahid and Raheel (2019), exponentiated discrete inverse Rayleigh (EDIR) presented by Mashhadzadeh and Mirmostafaee (2020), DIKum introduced by

EL-Helbawy *et al.* (2022) and *two parameter discrete Lindley* (TDL) distribution considered by Hussain *et al*. (2016).

The comparison is done by using K-S statistic, the corresponding p-value and other criteria for the purpose of model selection including Akaike information criterion (AIC), Akaike information criterion with correction (AICC) and Bayesian information criterion (BIC), where

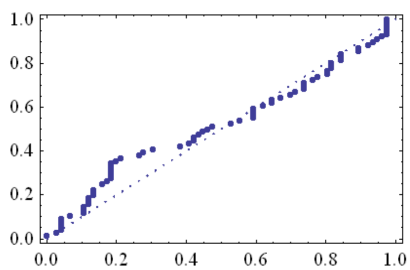
$AIC = 2k - 2(L)$,

$AICC = AIC + 2\,(k + 1)\; n - k - 1$,

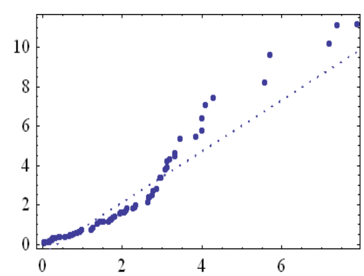
and

$BIC = (n) - 2log(L)$,

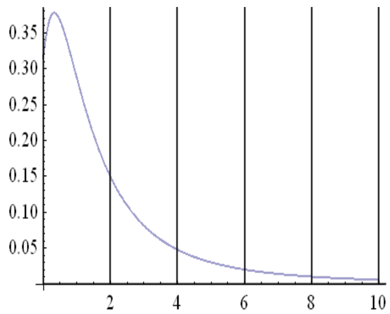
where k denotes the number of the estimated parameters, L is the maximized value of the likelihood function for the estimated model, and n is the sample size. The distribution which has the lowest values of AIC, AICC, BIC and the highest p-value, fits better to the real data set.



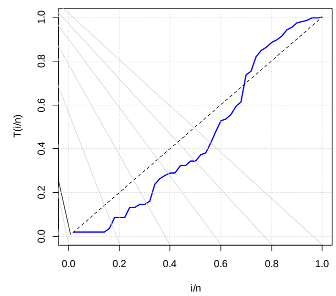
PP-plot



QQ-plot

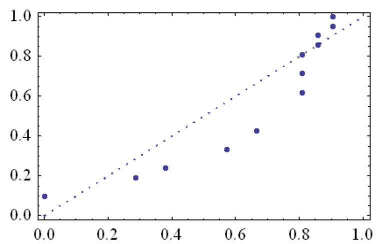


Fitted pmf

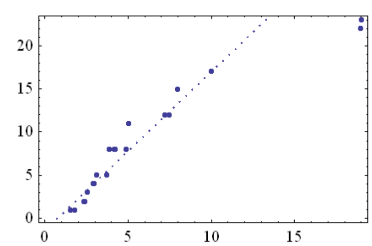


TTT-Plot

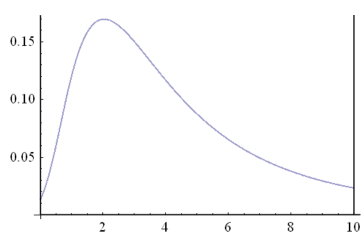
Figure4: The PP-plot, QQ-plot, fitted pmf and TTT-plot of DAIKum distribution for the first data set



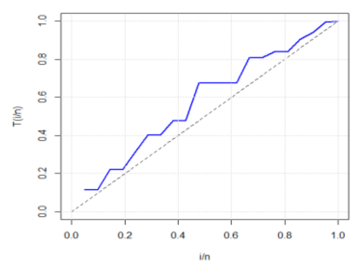
PP-plot



QQ-plot



Fitted pdf



TTT-Plot

Figure5: The PP-plot, QQ-plot, fitted pmf and TTT-plot of DAIKum distribution for the second data set

5.3 Concluding Remarks

1. It is observed that the ML averages of the estimates for the parameters α, λ and β, perform better when the sample size *n* increases. Comparing the results of the variance, RABs and REs when *n*=30 with the corresponding results when *n*=60 and *n*=120 under the same level of censoring and the same population parameter values of α, λ and β, ensured the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

2. Regarding the results of the variance, RABs and REs, the averages of the estimates for sf, hrf and ahrf seem to perform better (decrease) when the sample size *n* increases.

3. The two-sided 95% confidence intervals for the parameters of DAIKum distribution become narrower as the sample size increases.

4. The previous remarks are expected since decreasing the level of censoring means that more information is provided by the sample and hence increasing the accuracy of the estimates.

5. The *total time test* (TTT) plot can be used to get information about the shape of hrf of the given data set, which helps in selecting a particular model to fit a provided data.

   Figures 4 and 5, display the TTT plots of the two real data sets which indicate that the hrf has several shapes.

Moreover, the fitted pmf, PP and QQ plots ensured that the DAIKum distribution fit the two real data sets.

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Table2.

Averages, relative absolute biases, relative errors, variances of ML estimates and 95% confidence intervals of the parameters from DAIKum distribution for different sample sizes n, censoring size r and the replications NR= 5000

($\alpha = 2$, $\lambda = 5$ *and* β=50)

n	r	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	18	α β λ	1.9195 51.6607 4.2719	0.0402 0.0332 0.1456	0.0496 0.0372 0.1531	0.0033 0.6954 0.0558	2.0329 53.2951 4.7350	1.8062 50.0262 3.8088	0.2267 3.2689 0.9263
	24	α β λ	1.9661 50.9404 4.6435	0.0170 0.0188 0.0713	0.0211 0.0249 0.0767	0.0006 0.6615 0.0198	2.0152 52.5346 4.9194	1.9170 49.3463 4.3675	0.0982 3.1883 0.5519
	30	α β λ	1.9778 50.7309 4.7089	0.0111 0.0146 0.0582	0.0116 0.0156 0.0591	0.00005 0.0717 0.0027	1.991 51.2556 4.8111	1.9645 50.2062 4.6067	0.0266 1.0494 0.2044
60	36	α β λ	1.9273 51.5762 4.2956	0.0363 0.0315 0.1408	0.0378 0.0320 0.1422	0.0004 0.2227 0.0091	1.9685 52.1424 4.4830	1.8862 51.0100 4.1082	0.0823 1.1325 0.3747
	48	α β λ	1.9691 50.8171 4.6684	0.0155 0.0163 0.0663	0.0160 0.0189 0.0676	0.00007 0.1473 0.0043	1.9855 51.6421 4.7966	1.9527 50.4921 4.5402	0.0328 1.1500 0.2564
	60	α β λ	1.9793 50.7078 4.7152	0.0104 0.0142 0.0571	0.0105 0.0144 0.0572	0.00001 0.0171 0.0008	1.9858 50.9643 4.7697	1.9727 50.4514 4.6608	0.0131 0.5130 0.1088
120	72	α β λ	1.9588 50.7809 4.6126	0.0206 0.0156 0.0775	0.0212 0.01778 0.0783	0.00010 0.0835 0.0030	1.9780 50.9942 4.7198	1.9397 50.5675 4.5053	0.0383 0.4267 0.2145
	96	α β λ	1.9701 50.7034 4.6837	0.0149 0.0141 0.0633	0.0151 0.0160 0.0640	0.00002 0.0120 0.0024	1.9797 50.9556 4.7789	1.9606 50.6512 4.5885	0.0192 0.3044 0.1904
	120	α β λ	1.9792 50.6682 4.7158	0.0103 0.0133 0.0568	0.0104 0.0134 0.0570	0.000007 0.0068 0.0007	1.9845 50.8364 4.7681	1.9739 50.5354 4.6635	0.0106 0.3010 0.1046

Table3.
Averages, relative absolute biases, relative errors, variances of ML estimates and 95% confidence intervals of the parameters from DAIKum distribution for different sample sizes n, censoring size r and the replications NR= 5000

($\alpha = 2$, β=50, $\lambda = 5$)

n	r	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	18	$R(t_0)$	0.9503	0.1209	0.1223	0.0003	0.9809	0.9196	0.0614
		$h(t_0)$	0.5190	0.2733	0.2957	0.0065	0.6771	0.3609	0.3162
		$ah\ (t_0)$	0.7428	0.4069	0.4219	0.0196	1.0171	0.4686	0.5484
	24	$R(t_0)$	0.9073	0.0703	0.0726	0.0002	0.9374	0.8772	0.0602
		$h(t_0)$	0.6282	0.1203	0.1364	0.0021	0.7180	0.5384	0.1795
		$ah(t_0)$	0.9948	0.2056	0.2193	0.0091	1.1820	0.8077	0.3742
	30	$R(t_0)$	0.9008	0.0626	0.0633	0.00006	0.9159	0.8858	0.0301
		$h(t_0)$	0.6429	0.0998	0.1016	0.0002	0.6694	0.6165	0.0529
		$ah\ (t_0)$	1.0305	0.1772	0.1796	0.0014	1.1028	0.9581	0.1447
60	36	$R(t_0)$	0.9506	0.1202	0.1217	0.0002	0.9692	0.9319	0.0373
		$h(t_0)$	0.5282	0.2604	0.2638	0.0009	0.5877	0.4687	0.1190
		$ah\ (t_0)$	0.7533	0.3985	0.4017	0.0041	0.8786	0.6279	0.2507
	48	$R(t_0)$	0.9045	0.0669	0.0680	0.0001	0.9246	0.8844	0.0402
		$h(t_0)$	0.6361	0.1093	0.1124	0.0003	0.6728	0.5995	0.0733
		$ah(t_0)$	1.0122	0.1918	0.1958	0.0024	1.1088	0.9155	0.1933
	60	$R(t_0)$	0.8977	0.0590	0.0592	0.00002	0.9059	0.8895	0.0164
		$h(t_0)$	0.6482	0.0924	0.0929	0.00005	0.6619	0.6345	0.0275
		$ah\ (t_0)$	1.0449	0.1657	0.1664	0.0004	1.0836	1.0062	0.0775
120	72	$R(t_0)$	0.9121	0.0759	0.0764	0.000075	0.9269	0.8972	0.0297
		$h(t_0)$	0.6224	0.1285	0.1301	0.0002	0.6510	0.5938	0.0572
		$ah\ (t_0)$	0.9746	0.2218	0.2239	0.0015	1.0494	0.8998	0.1497
	96	$R(t_0)$	0.9020	0.0640	0.0647	0.00007	0.9178	0.8862	0.0316
		$h(t_0)$	0.6406	0.1031	0.1049	0.0002	0.6677	0.6134	0.0543
		$ah(t_0)$	1.0240	0.1824	0.1848	0.0014	1.0973	0.9506	0.1466
	120	$R(t_0)$	0.8970	0.0581	0.0583	0.00001	0.9036	0.8905	0.0131
		$h(t_0)$	0.6494	0.0907	0.0910	0.00003	0.6602	0.6386	0.0217
		$ah\ (t_0)$	1.0482	0.1630	0.1635	0.0002	1.0789	1.0174	0.0615

Table4.
Averages, relative absolute biases, relative errors, variances of ML estimates and 95% confidence intervals of the parameters from DAIKum distribution for different sample sizes n and the replications NR= 2000

n	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	$\alpha = 2$ $\beta = 5$ $\lambda = 1$	2.0260 6.3105 1.0375	0.0130 0.2621 0.0376	0.2207 0.3310 0.1566	0.1941 1.0215 0.0231	2.8896 8.2914 1.3355	1.1623 4.3296 0.7396	1.7272 3.9618 0.5959
	$\alpha = 2$ $\beta = 20$ $\lambda = 1$	1.9873 22.2076 1.0076	0.0063 0.1103 0.0077	0.1984 0.2526 0.0958	0.1573 20.6581 0.0091	2.7648 31.1160 1.1948	1.2099 13.2992 0.8204	1.5549 17.8169 0.3745
	$\alpha = 2$ $\beta = 50$ $\lambda = 1$	1.9873 51.9830 0.9949	0.0063 0.0397 0.0050	0.2070 0.2090 0.0691	0.1712 105.3284 0.0047	2.7984 72.0984 1.1299	1.1762 31.8676 0.8599	1.6222 40.2308 0.2700
60	$\alpha = 2$ $\beta = 5$ $\lambda = 1$	1.9437 6.2132 1.0288	0.0282 0.2426 0.0288	0.1976 0.2872 0.1055	0.1530 0.5897 0.0103	2.7104 7.7183 1.2277	1.1769 4.7079 0.8300	1.5334 3.0104 0.3976
	$\alpha = 2$ $\beta = 20$ $\lambda = 1$	1.9891 22.4297 1.0155	0.0054 0.1215 0.0155	0.2005 0.2380 0.0864	0.1607 16.7557 0.0072	2.7748 30.4527 1.1820	1.2034 14.4066 0.8489	1.5714 16.0460 0.3331
	$\alpha = 2$ $\beta = 50$ $\lambda = 1$	1.9785 52.3870 0.9989	0.0107 0.0477 0.0011	0.1863 0.1875 0.0560	0.1384 82.2408 0.0031	2.7077 70.1616 1.1087	1.2493 34.6124 0.8890	1.4584 35.5492 0.2195
120	$\alpha = 2$ $\beta = 5$ $\lambda = 1$	1.8572 6.2551 1.0233	0.0714 0.2510 0.0233	0.1716 0.2807 0.0669	0.0974 0.3948 0.0039	2.4689 7.4866 1.1463	1.2455 5.0237 0.9003	1.2234 2.4629 0.2459
	$\alpha = 2$ $\beta = 20$ $\lambda = 1$	1.9559 22.2354 1.0134	0.0220 0.1118 0.0134	0.1960 0.1898 0.0657	0.1518 9.4168 0.0041	2.7196 28.2499 1.1396	1.1922 16.2207 0.8872	1.5274 12.0292 0.2523
	$\alpha = 2$ $\beta = 50$ $\lambda = 1$	1.98167 52.6543 1.0023	0.00916 0.0531 0.0023	0.1814 0.1612 0.0476	0.1312 57.9273 0.0023	2.6917 67.5718 1.0956	1.2716 37.7367 0.9091	1.4201 29.8351 0.1865

Table5.
Averages, relative absolute biases, relative errors, variances of ML estimates and 95% confidence intervals of the parameters from DAIKum distribution for different sample sizes n and the replications NR= 2000

n	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	$\alpha=5$ $\beta=5$ $\lambda=3$	5.8631 6.3110 2.2656	0.1726 0.2622 0.2448	0.1854 0.2772 0.2570	0.1142 0.2028 0.0552	6.5254 7.1937 2.7260	5.2009 5.4283 1.8052	1.3245 1.7654 0.9208
	$\alpha=5$ $\beta=20$ $\lambda=3$	5.5669 24.4722 2.5491	0.1134 0.2236 0.1503	0.1296 0.2329 0.1668	0.0990 1.6977 0.0472	6.1837 27.0259 2.9751	4.9501 21.9184 2.1229	1.2335 5.1075 0.8522
	$\alpha=5$ $\beta=50$ $\lambda=3$	5.3201 59.9424 2.7192	0.0640 0.1988 0.0936	0.1030 0.2129 0.1141	0.1629 14.4681 0.0382	6.1114 67.3977 3.1025	4.5288 52.4872 2.3358	1.5826 14.9105 0.7666
60	$\alpha=5$ $\beta=5$ $\lambda=3$	5.8701 6.3426 2.1799	0.1740 0.2685 0.2733	0.1808 0.2766 0.2811	0.0599 0.1099 0.0384	6.3498 6.9924 2.5641	5.3903 5.6929 1.7956	0.9595 1.2994 0.7685
	$\alpha=5$ $\beta=20$ $\lambda=3$	5.5526 24.4480 2.5552	0.1105 0.2224 0.1482	0.1187 0.2268 0.1572	0.0468 0.8037 0.0249	5.9769 26.2051 2.8643	5.1283 22.6908 2.2461	0.8487 3.5143 0.6182
	$\alpha=5$ $\beta=50$ $\lambda=3$	5.2852 59.6181 2.7175	0.0570 0.1924 0.0942	0.0777 0.1998 0.1047	0.0697 7.3172 0.0188	5.8026 64.9199 2.9868	4.7678 54.3163 2.4482	1.0348 10.6036 0.5386
120	$\alpha=5$ $\beta=5$ $\lambda=3$	5.8993 6.3949 2.1270	0.1799 0.2789 0.2909	0.1827 0.2821 0.2953	0.0257 0.0440 0.0227	6.2136 6.8065 2.4225	5.5849 5.9835 1.8315	0.6286 0.8230 0.5910
	$\alpha=5$ $\beta=20$ $\lambda=3$	5.5322 24.3769 2.5518	0.1064 0.2188 0.1494	0.1107 0.2211 0.1542	0.0231 0.3962 0.0131	5.8302 25.6106 2.7763	5.2343 23.1433 2.3275	0.5958 2.4674 0.4488
	$\alpha=5$ $\beta=50$ $\lambda=3$	5.2747 59.4791 2.7193	0.0549 0.1896 0.0936	0.0653 0.1930 0.0987	0.0312 3.3568 0.0089	5.6213 63.0701 2.9044	4.9281 55.8880 2.5342	0.6932 7.1821 0.3702

Table6.

ML estimates for the parameters, sf, hrf and alternative hrf and their standard errors for the real data based on Type II censoring

Real Data	*n*	Parameters	Estimates	SE
Application I	76	α λ β $R(t_0)$ $h(t_0)$ $alt\ h\ (t_0)$	2.8159 1.7433 4.1108 0.8500 0.2849 0.3354	0.1971 0.2055 0.2034 0.2195 0.2302 0.2292
Application II	21	α λ β $R(t_0)$ $h(t_0)$ $alt\ h\ (t_0)$	3.0969 1.5535 12.9282 0.9975 0.0398 0.0407	0.6409 0.6812 0.6096 0.6961 0.7222 0.7221

Table 7: Goodness-of-fit measures for fitted models of real data set I

Model	Estimator MLE(SE)				-2LL	AIC	BIC	CAIC	K-S P-Value
DAIKum	α 2.815 (0.1971)	λ 1.7433 (0.2055)	β 4.1108 (0.2034)		140.647	146.647	153.64	146.981	0.0921 0.9061
DIKum	λ 1.7934 (0.2049)	β 6.6871 (0.2494)			311.665	315.665	320.326	315.829	0.1710 0.2166
GDW	α 2.6777 (0.1975)	θ 2.0580 (0.2019)	λ 0.6012 (0.2241)	β 1.2778 (0.2122)	374.05	382.05	391.373	382.613	0.2105 0.0686
DGIW	α 2.1658 (0.2008)	λ 0.3687 (0.2285)	β 2.0380 (0.0.2021)		295.137	301.137	308.129	301.47	0.1842 0.1520
DIW	λ 0.4284 (0.2274)	β 0.8614 (0.2193)			357.452	361.452	366.113	361.616	0.1578 0.3011
DMOW	α 4.3815 (0.2247)	λ 0.5694 (0.2247)	β 1.1432 (0.2144)		355.922	361.922	368.915	362.256	0.1973 0.1034
DMOGE	α 0.7964 (0.2205)	λ 0.7174 (0.2219)	β 0.8044 (0.2203)		361.457	367.457	374.449	367.79	0.2105 0.0686
DMIR	λ 0.3116 (0.2296)	β 0.9342 (0.2180)			347.698	351.698	356.359	351.862	0.1973 0.1033
EDIR	λ 0.3554 (0.2288)	β 0.6130 (0.2239)			332.45	336.45	341.112	336.615	0.1710 0.2170
TDL	λ 0.5888 (0.2243)	β 1.2170 (0.2132)			328.753	332.753	337.415	332.917	0.1710 0.2166

Table 8: Goodness-of-fit measures for fitted models of real data set II

Model	Estimator MLE(SE)				-2LL	AIC	BIC	CAIC	K-S P-Value
DAIKum	α 3.0969 (0.6409)	λ 1.5535 (0.6812)	β 12.9283 (0.996)		72.5716	78.5716	81.7051	79.9833	0.2380 0.5860
DIKum	λ 1.0862 (1.0862)	β 4.9978 (0.5968)			139.039	143.039	145.128	143.706	0.2857 0.3370
GDW	α 1.5481 (0.6813)	θ 1.8091 (0.6743)	λ 0.9126 (0.6984)	β 1.2264 (0.6864)	131.935	139.935	144.113	142.435	0.3333 0.1759
DGIW	α 0.7438 (0.7030)	λ 0.3249 (0.7144)	β 2.4640 (0.6512)		147.48	153.48	156.614	154.892	0.3333 0.1650
DIW	λ 0.8866 (0.6991)	β 0.3133 (0.7147)			179.824	183.824	185.913	184.491	0.3809 0.0866
DMOW	α 2.3877 (0.6591)	λ 0.9010 (0.6981)	β 1.2762 (0.6886)		604.02	610.02	617.012	610.353	0.3809 0.0833
DMOGE	α 0.7372 (0.7032)	λ 0.9084 (0.6985)	β 0.4960 (0.7097)		148.945	154.945	158.079	156.357	0.3333 0.1730
DMIR	λ 0.2.3748 (0.6595)	β 0.10.8844 (0.5713)			139.989	143.989	146.078	144.656	0.2857 0.3399
EDIR	λ 0.3042 (0.7150)	β 0.1995 (0.7178)			153.116	157.116	159.205	157.782	0.3333 0.1767
TDL	λ 0.8461 (0.7002)	β 6.2672 (0.5740)			133.951	137.951	140.04	138.618	0.3809 0.0850